A NOETHER-DEURING THEOREM FOR DERIVED CATEGORIES

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Abstract. We prove a Noether-Deuring theorem for the derived category of bounded complexes of modules over a Noetherian algebra.

Introduction

The classical Noether-Deuring theorem states that given an algebra A over a field K and a finite extension field L of K, two A-modules M and N are isomorphic as A-modules, if $L \otimes_K M$ is isomorphic to $L \otimes_K N$ as an $L \otimes_K A$-module. In 1972 Roggenkamp gave a nice extension of this result to extensions S of local commutative Noetherian rings R and modules over Noetherian R-algebras.

For the derived category of A-modules no such generalisation was documented before. The purpose of this note is to give a version of the Noether-Deuring theorem, in the generalised version given by Roggenkamp, for right bounded derived categories of A-modules. If there is a morphism $\alpha \in Hom_{D(\Lambda)}(X, Y)$, then it is fairly easy to show that for a faithfully flat ring extension S over R the fact that $id_S \otimes \alpha$ is an isomorphism implies that α is an isomorphism. This is done in Proposition 1. More delicate is the question if only an isomorphism in $Hom_{D(S\otimes_R\Lambda)}(S \otimes_R X, S \otimes_R Y)$ is given. Then we need further finiteness conditions on Λ and on R and proceed by completion of R and then a classical going-down argument. This is done in Theorem 4 and Corollary 8.

For the notation concerning derived categories we refer to Verdier [5]. In particular, $D(A)$ (resp $D^-(A)$, resp $D^b(A)$) denotes the derived category of complexes (resp. right bounded complexes, resp. bounded complexes) of finitely generated A-modules, $K^-(A-proj)$ (resp. $K^b(A-proj)$, resp $K^{-,b}(A-proj)$) is the homotopy category of right bounded complexes (resp. bounded complexes, resp. right bounded complexes with bounded homology) of finitely generated projective A-modules. For a complex Z we denote by $H_i(Z)$ the homology of Z in degree i, and by $H(Z)$ the graded module given by the homology of Z.

1. The result

We start with an easy observation.

Proposition 1. *Let R be a commutative ring and let Λ be an R-algebra. Let S be a commutative faithfully flat R-algebra. Denote by $D(\Lambda)$ the derived category of complexes of finitely generated Λ-modules. Then if there is $\alpha \in Hom_{D(\Lambda)}(X, Y)$ so that $id_S \otimes^{\mathbb{L}}_R \alpha \in Hom_{D(S\otimes_R\Lambda)}(S \otimes^{\mathbb{L}}_R X, S \otimes^{\mathbb{L}}_R Y)$ is an isomorphism in $D(S \otimes_R \Lambda)$, then α is an isomorphism in $D(\Lambda)$.*

Proof. Let Z be a complex in $D(\Lambda)$. Since S is flat over R the functor $S \otimes_R - : R - Mod \longrightarrow S - Mod$ is exact, and hence the left derived functor $S \otimes^{\mathbb{L}}_R -$ coincides with the ordinary tensor product functor $S \otimes_R -$. We can therefore work with the usual tensor product and a complex Z of Λ-modules.

We claim that since S is flat, $S \otimes_R -$ induces an isomorphism $S \otimes_R H(Z) \simeq H(S \otimes^{\mathbb{L}}_R Z)$. If ∂_Z is the differential of Z, then

$$0 \longrightarrow \ker(\partial_Z) \longrightarrow Z \xrightarrow{\partial_Z} im(\partial_Z) \longrightarrow 0$$

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is exact in the category of Λ-modules.

Since S is flat,

$$0 \longrightarrow S \otimes_R \ker(\partial_Z) \longrightarrow S \otimes_R Z \overset{id_S \otimes_R \partial_Z}{\longrightarrow} S \otimes_R im(\partial_Z) \longrightarrow 0$$

is exact. Hence

$$\ker(id_S \otimes_R \partial_Z) = S \otimes_R \ker(\partial_Z) \text{ and } im(id_S \otimes_R \partial_Z) = S \otimes_R im(\partial_Z).$$

This shows the claim.

Since $id_S \otimes_R \alpha$ is an isomorphism, its cone $C(id_S \otimes_R \alpha)$ is acyclic. Moreover, $C(id_S \otimes_R \alpha) = S \otimes_R C(\alpha)$ by the very construction of the mapping cone. But now,

$$0 = H(C(id_S \otimes_R \alpha)) = H(S \otimes_R C(\alpha)) = S \otimes_R H(C(\alpha)).$$

Since S is faithfully flat, this implies $H(C(\alpha)) = 0$ and therefore $C(\alpha)$ is acyclic. We conclude that α is an isomorphism in $D(\Lambda)$ which shows the statement. ■

Remark 2. Observe that we assumed that $X \xrightarrow{\alpha} Y$ is assumed to be a morphism in $D(\Lambda)$. The question if the existence of an isomorphism $S \otimes_R X \xrightarrow{\hat{\alpha}} S \otimes_R Y$ in $D(S \otimes_R \Lambda)$ implies the existence of a morphism $\alpha : X \longrightarrow Y$ in $D(\Lambda)$ so that $id_S \otimes_R^{\mathbb{L}} \alpha$ is an isomorphism is left open. Under stronger hypotheses this is the purpose of Theorem 4 below. The proof follows [4] which deals with the module case.

Lemma 3. *If S is a faithfully flat R-module and Λ is a Noetherian R-algebra, then for all objects X and Y of $D^-(\Lambda)$ we get*

$$Hom_{D^-(S \otimes_R \Lambda)}(S \otimes_R X, S \otimes_R Y) \simeq S \otimes_R Hom_{D^-(\Lambda)}(X, Y).$$

Proof. We use the equivalence of categories $K^-(\Lambda - proj) \simeq D^-(\Lambda)$ and suppose therefore that X and Y are right bounded complexes of finitely generated projective Λ-modules. We even may assume that X and Y are right bounded complexes of finitely generated free Λ-modules. But

$$S \otimes_R Hom_\Lambda(\Lambda^n, U) = S \otimes_R U^n = (S \otimes_R U)^n = Hom_{S \otimes_R \Lambda}((S \otimes_R \Lambda)^n, S \otimes_R U)$$

which proves the statement. ■

Theorem 4. *Let R be a commutative Noetherian ring, let S be a commutative Noetherian R-algebra and suppose that S is faithfully flat as R-module. Suppose $S \otimes_R rad(R) = rad(S)$. Let Λ be a Noetherian R-algebra, let X and Y be two objects of of $D^-(\Lambda)$ and suppose that $End_{D^-(X)}(X)$ is a finitely generated R-module. Then*

$$S \otimes_R^{\mathbb{L}} X \simeq S \otimes_R^{\mathbb{L}} Y \Leftrightarrow X \simeq Y.$$

Remark 5. We observe that if R is semilocal and $S = \hat{R}$ is the $rad(R)$-adic completion, then S is faithfully flat as R-module and $S \otimes_R rad(R) = rad(S)$.

Proof of Theorem 4. According to the hypotheses we now suppose that $End_{D^-(\Lambda)}(X)$ is a finitely generated R-module and that $S \otimes_R rad(R) = rad(S)$. We only need to show "$\Rightarrow$" and assume therefore that X and Y are in $K^-(\Lambda - proj)$, and that $S \otimes_R X$ and $S \otimes_R Y$ are isomorphic.

Let $X_S := S \otimes_R X$ and $S \otimes_R Y =: Y_S$ in $D^-(S \otimes_R \Lambda)$ to shorten the notation. If X_S is a direct factor of Y_S, then the mapping

$$\varphi_S := \sum_{i=1}^{n} s_i \otimes \varphi_i : X_S \longrightarrow Y_S$$

for $s_i \in S$ and $\varphi_i \in Hom_{D^-(\Lambda)}(X, Y)$ has a left inverse $\psi : Y_S \longrightarrow X_S$:

$$\psi \circ \varphi_S = id_{X_S}.$$

Then,

$$0 \longrightarrow rad(R) \longrightarrow R \longrightarrow R/rad(R) \longrightarrow 0$$

is exact and since S is flat over R we get that

$$0 \longrightarrow S \otimes_R rad(R) \longrightarrow S \longrightarrow S \otimes_R (R/rad(R)) \longrightarrow 0$$

is exact. This shows

$$S \otimes_R (R/rad(R)) \simeq S/(S \otimes_R rad(R)).$$

By hypothesis we have $S \otimes_R rad(R) = rad(S)$, identifying canonically $S \otimes_R R \simeq S$. Then there are $r_i \in R$ so that $1_S \otimes r_i - s_i \in rad(S)$ for all $i \in \{1, \dots, n\}$.

Put

$$\varphi := \sum_{i=1}^{n} r_i \varphi_i \in Hom_{D^-(\Lambda)}(X, Y).$$

Then

$$\begin{aligned}\sum_{i=1}^{n} \psi \circ (1_S \otimes (r_i\varphi_i)) - 1_S \otimes id_X &= \sum_{i=1}^{n} (\psi \circ (1_S \otimes r_i\varphi_i) - \psi \circ (s_i \otimes \varphi_i)) \\ &= \psi \circ \left(\sum_{i=1}^{n} (1_S \otimes r_i - s_i)\,(id_S \otimes \varphi_i) \right) \\ &\in \big(rad(S) \otimes_R End_{D^-(\Lambda)}(X)\big)\end{aligned}$$

and since $End_{D^-(\Lambda)}(X)$ is a Noetherian R-module, using Nakayama's lemma we obtain that $\psi \circ (\sum_{i=1}^{n} 1_S \otimes r_i\varphi_i)$ is invertible in $S \otimes_R End_{D^-(\Lambda)}(X)$. Hence $id_S \otimes_R \varphi$ is left split and therefore

$$X_S \overset{id_S \otimes_R \varphi}{\longrightarrow} Y_S \longrightarrow C(id_S \otimes_R \varphi) \xrightarrow{0} X_S[1]$$

is a distinguished triangle, with $C(id_S \otimes_R \varphi)$ being the cone of $id_S \otimes_R \varphi$. However,

$$C(id_S \otimes_R \varphi) = S \otimes_R C(\varphi)$$

and hence

$$X_S \overset{id_S \otimes_R \varphi}{\longrightarrow} Y_S \longrightarrow S \otimes_R C(\varphi) \xrightarrow{0} X_S[1]$$

is a distinguished triangle. Since

$$X \xrightarrow{\varphi} Y \longrightarrow C(\varphi) \xrightarrow{\beta} X[1]$$

is a distinguished triangle, and tensoring with the flat extension S over R gives a distinguished triangle as well, we get $id_S \otimes \beta = 0$.

Since φ_S is an isomorphism, φ_S has a right inverse $\chi : Y_S \longrightarrow X_S$ as well. Now, since $X_S \simeq Y_S$, since S is faithfully flat over R, and since $End_{D^-(\Lambda)}(X)$ is finitely generated as R-module, using Lemma 3 we obtain that $End_{D^-(\Lambda)}(Y)$ is finitely generated as R-module as well. The same argument as for the left inverse ψ shows that $(id_S \otimes \varphi) \circ \chi$ is invertible in $S \otimes_R End_{D^-(\Lambda)}(Y)$. Hence

$$X_S \overset{id_S \otimes_R \varphi}{\longrightarrow} Y_S \xrightarrow{0} S \otimes_R C(\varphi) \xrightarrow{0} X_S[1]$$

is a distinguished triangle. This shows that $S \otimes_R C(\varphi)$ is acyclic, and hence

$$0 = H(S \otimes_R C(\varphi)) = S \otimes_R H(C(\varphi)).$$

Since S is faithfully flat over R also $H(C(\varphi)) = 0$, which implies that $C(\varphi)$ is acyclic and therefore φ is an isomorphism.

This proves the theorem. ■

Let A be an algebra over a complete discrete valuation ring R which is finitely generated as modules over R. We shall need a Krull-Schmidt theorem for the derived category of bounded complexes over A. This fact seems to be well-known. For the convenience of the reader we give a short proof.

Proposition 6. *Let R be a commutative ring and let A be an R-algebra. If Fitting's lemma holds for finitely generated projective A-modules, then the Krull-Schmidt theorem holds for $K^b(A-proj)$ as well.*

Proof. By Jensen-Su-Zimmermann [2, Lemma 2] we get that for two complexes X and Y so that $X \simeq Y$ in $K^b(A-proj)$, we may suppose that X and Y are isomorphic as graded modules (possibly in $K^-(A-proj)$, but since X and Y are both objects of $K^b(A-proj)$, it is immediate to see that one may actually restrict to bounded complexes of projective modules). By [2, Lemma 1] we get that if $X \simeq Y$ in $K^b(A-proj)$, and if X and Y are isomorphic as graded modules, then actually $X \simeq Y$ as complexes. We may therefore work in the abelian category of complexes. Let u be an endomorphism of X. Then $X = X' \oplus X''$ as graded modules, by Fitting's lemma. Recall the construction of X' and X'' in the proof of Fitting's lemma as $X' = im(u^N)$ and $X'' = \ker(u^N)$ for some large $N \in \mathbb{N}$. Here u is a morphism of complexes, and hence X' and X'' are both subcomplexes of X. Moreover, X' and X'' are both projective modules, since X is projective. The restriction of u on X' is an automorphism and the restriction of u on X'' is nilpotent. This shows that Fitting's lemma holds as well in the category of bounded complexes of projective modules. Hence, the endomorphism ring of an indecomposable object is local and the Krull-Schmidt theorem is an easy consequence by the classical proof as in [3]. This shows the proposition. ■

We obtain the following consequence.

Proposition 7. *Let A be a Noetherian ring so that $A-proj$ admits Fitting's lemma. Then any complex X of $D^b(A)$ admits a unique decomposition $X \simeq \bigoplus_{i\in I} X_i$ into indecomposable complexes X_i and the decomposition is unique up to isomorphism and permutation of factors.*

Proof. Since A is Noetherian, $D^b(A) \simeq K^{-,b}(A-proj)$ and given a bounded complex X, it is isomorphic to a complex $Y := P_X$ in $K^{-,b}(A-proj)$. Since the homology of Y is bounded, we may find N_0 so that $H_n(Y) = 0$ if $|n| \geq N_0$. We cut the complex Y in degree N_0, in the sense that there is a complex $\sigma_{N_0}Y$ given by $(\sigma_{N_0}Y)_m = Y_m$ if $m \leq N_0$ and $(\sigma_{N_0}Y)_m = 0$ if $m > N_0$. For $m \leq N_0$ the degree m differential of $\sigma_{N_0}Y$ is the same as the differential of Y, and is 0 in larger degrees.

$$\begin{tikzcd}
\ker(\partial_{N_0})[N_0+1] & : & \ldots \arrow[r, "\partial_{N_0+2}"] & P_{N_0+1} \arrow[r] & 0 \arrow[r] & \ldots \arrow[r] & 0 \arrow[r] & 0 \\
Y & : & \ldots \arrow[r, "\partial_{N_0+2}"] & P_{N_0+1} \arrow[r, "\partial_{N_0+1}"] \arrow[u] & P_{N_0} \arrow[r, "\partial_{N_0}"] \arrow[u] & \ldots \arrow[r, "\partial_{m+1}"] & P_m \arrow[r] \arrow[u] & 0 \arrow[u] \\
\sigma_{N_0}Y & : & \ldots \arrow[r] & 0 \arrow[r] \arrow[u] & P_{N_0} \arrow[r, "\partial_{N_0}"] \arrow[u] & \ldots \arrow[r, "\partial_{m+1}"] & P_m \arrow[r] \arrow[u] & 0 \arrow[u]
\end{tikzcd}$$

which gives the distinguished triangle

$$\sigma_{N_0}Y \longrightarrow Y \longrightarrow \ker(\partial_{N_0})[N_0+1] \xrightarrow{\partial_{N_0+1}} \sigma_{N_0}Y[1].$$

The choice of N_0 implies that $H_{N_0}(\sigma_{N_0}Y) = \ker(\partial_{N_0})$. By Proposition 6 we see that $\sigma_{N_0}Y$ decomposes

$$\sigma_{N_0}Y \simeq Q_1 \oplus \cdots \oplus Q_s$$

uniquely into a direct sum of indecomposable objects Q_i for $i \in \{1, \ldots, s\}$. The decomposition is induced by idempotent endomorphisms $e_1, \ldots, e_s$ of $\sigma_{N_0}Y$ and hence the idempotent endomorphisms $e_1, \ldots, e_s$ of $\sigma_{N_0}Y$ induce idempotent endomorphisms $H_{N_0}(e_1), \ldots, H_{N_0}(e_s)$ of $H_{N_0}(\sigma_{N_0}Y)$. Therefore

$$\ker(\partial_{N_0}) = \bigoplus_{i=1}^{s} (H_{N_0}(e_i))(\ker(\partial_{N_0}))$$

and the idempotent endomorphisms e_i of $\sigma_{N_0}Y$ extend to idempotent endomorphisms $\hat{e}_i$ of the initial object Y.

A maybe more direct way of seeing this is to observe that the decomposition $\sigma_{N_0}Y \simeq Q_1 \oplus \cdots \oplus Q_s$ induces a decomposition $\partial_{N_0} = \bigoplus_{i=1}^s \partial^i_{N_0}$ for the corresponding differential ∂^i of Q_i. Hence

$$\ker(\partial_{N_0}) = \ker\left(\bigoplus_{i=1}^s \partial^i_{N_0}\right) = \bigoplus_{i=1}^s \ker\left(\partial^i_{N_0}\right)$$

and Y in degrees bigger than N_0 is a projective resolution of the module $\bigoplus_{i=1}^s \ker(\partial^i_{N_0})$, whence a direct sum of projective resolutions of the modules $\ker(\partial^i_{N_0})$ for all $i \in \{1, \dots, s\}$.

Hence, the Krull-Schmidt theorem holds for $D^b(A)$ as well. ■

For the next Corollary we follow closely [4].

Corollary 8. *Let R be a commutative Noetherian ring, let S be a commutative R-algebra so that $\hat{S} := \hat{R} \otimes_R S$ is a faithful projective $\hat{R}$-module of finite type. Let Λ be a Noetherian R-algebra, finitely generated as R-module, and let X and Y be two objects of of $D^b(\Lambda)$ and suppose that $End_{D^b(\Lambda)}(X)$ is a finitely generated R-module. Then*

$$S \otimes^{\mathbb{L}}_R X \simeq S \otimes^{\mathbb{L}}_R Y \Leftrightarrow X \simeq Y.$$

Proof. If $S \otimes^{\mathbb{L}}_R X \simeq S \otimes^{\mathbb{L}}_R Y$ in $D^b(S \otimes_R \Lambda)$, we get $\hat{S} \otimes^{\mathbb{L}}_R X \simeq \hat{S} \otimes^{\mathbb{L}}_R Y$ in $D^b(\hat{S} \otimes_R \Lambda)$. Since R is semilocal with maximal ideals $m_1, \dots, m_s$ we get $\hat{R} = \prod_{i=1}^s \hat{R}_{m_i}$ for the completion $\hat{R}_{m_i}$ of R at m_i. Now, $\hat{S}$ is projective faithful of finite type, and so there are $n_1, \dots, n_s$ with

$$\hat{S} \simeq \prod_{i=1}^s (\hat{R}_{m_i})^{n_i}$$

and therefore $\hat{S} \otimes^{\mathbb{L}}_R X \simeq \hat{S} \otimes^{\mathbb{L}}_R Y$ implies

$$\prod_{i=1}^s (\hat{R}_{m_i})^{n_i} \otimes^{\mathbb{L}}_R X \simeq \prod_{i=1}^s (\hat{R}_{m_i})^{n_i} \otimes^{\mathbb{L}}_R Y.$$

Hence

$$(\hat{R}_{m_i} \otimes^{\mathbb{L}}_R X)^{n_i} \simeq (\hat{R}_{m_i} \otimes^{\mathbb{L}}_R Y)^{n_i}$$

for each i, and therefore by Proposition 7

$$\hat{R}_{m_i} \otimes^{\mathbb{L}}_R X \simeq \hat{R}_{m_i} \otimes^{\mathbb{L}}_R Y$$

for each i. By Theorem 4 we obtain $X \simeq Y$. ■

We get cancellation of factors from this statement.

Corollary 9. *Under the hypothesis of Theorem 4 or of Corollary 8 we get $X \oplus U \simeq Y \oplus U$ in $D^b(\Lambda)$ implies $X \simeq Y$.*

Proof. This is clear by Corollary 8 in combination with Proposition 7. ■

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References

[1] Nicolas Bourbaki, ALGÈBRE COMMUTATIVE CHAPITRE III Hermann Paris 1959.

[2] Bernt Tore Jensen, Xiuping Su and Alexander Zimmermann, *Degenerations for derived categories*, Journal of Pure and Applied Algebra **198** (2005) 281-295

[3] Serge Lang, ALGEBRA, Addison-Wesley.

[4] Klaus W. Roggenkamp, *An extension of the Noether-Deuring theorem*, Proceedings of the American Mathematical Society **31** (1972) 423–426.

[5] Jean-Louis Verdier, DES CATÉGORIES DÉRIVÉES DES CATÉGORIES ABÉLIENNES, Astérisque **239** (1996).

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