ASYMPTOTIC HAUSDORFF DIMENSIONS OF CANTOR SETS ASSOCIATED WITH AN ASYMPTOTICALLY NON-HYPERBOLIC FAMILY

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ABSTRACT. The geometry of Cantor systems associated with an asymptotically non-hyperbolic family $(f_\epsilon)_{0\leq\epsilon\leq\epsilon_0}$ is studied in [Jia2]. By applying the geometry studied there, we prove that the Hausdorff dimension of the maximal invariant set of f_ϵ behaves like $1-K\epsilon^{\frac{1}{\gamma}}$ asymptotically, as was conjectured in [Jia1].

1. INTRODUCTION

During the last two decades, Cantor sets have played an important role in the study of chaotic dynamical systems. The geometry and the dimension of a Cantor set thus becomes interesting and important.

In the construction of a Cantor set on the real line $\mathbb{R}$, we remove infinitely many subintervals which are called the gaps of the Cantor set. The sizes and positions of these gaps determine the geometry and then the dimension of the Cantor set. We will show in this paper how the dimension is quantitatively determined by the geometry.

Key words and phrases. Asymptotically non-hyperbolic family, Cantor systems, Hausdorff dimension.

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Let $\mathcal{I} = \{\mathcal{I}_n\}_{n=0}^{\infty}$ be a sequence of families of disjoint, non-empty and compact intervals, and let $\mathcal{G} = \{\mathcal{G}_n\}_{n=1}^{\infty}$ be a sequence of families of disjoint, non-empty and open intervals.

Definition 1. *We call $\mathcal{C} = \{\mathcal{I}, \mathcal{G}\}$ a Cantor system if*

(1) *for each $0 \leq n < \infty$ and for each interval $I \in \mathcal{I}_n$, there is a unique interval G in $\mathcal{G}_{n+1}$ and two intervals L and R in $\mathcal{I}_{n+1}$ which lie to the left and to the right of G such that $I = L \cup G \cup R$ (see Figure 1);*

(2) *$\Lambda = \cap_{n=0}^{+\infty} \bigcup_{I \in \mathcal{I}_n} I$ is totally disconnected.*

The set Λ is called a Cantor set. We call each interval $I \in \mathcal{I}_n$ a n-level bridge and call each interval $G \in \mathcal{G}_n$ a n-level gap.

Definition 2. *The bridge geometry of $\mathcal{C}$ is the set of ratios*

$$\mathcal{BR} = \bigcup_{n=0}^{\infty} \Big\{ \frac{|J|}{|I|} \mid I = L \cup G \cup R \in \mathcal{I}_n \ \textit{with}\ J = L \ \textit{or}\ R \in \mathcal{I}_{n+1} \ \textit{and}\ G \in \mathcal{G}_{n+1} \Big\}.$$

The gap geometry of $\mathcal{C}$ is the set of ratios

$$\mathcal{GAP} = \bigcup_{n=0}^{\infty} \Big\{ \frac{|G|}{|J|} \mid I = L \cup G \cup R \in \mathcal{I}_n \ \textit{with}\ J = L \ \textit{or}\ R \in \mathcal{I}_{n+1} \ \textit{and}\ G \in \mathcal{G}_{n+1} \Big\}.$$

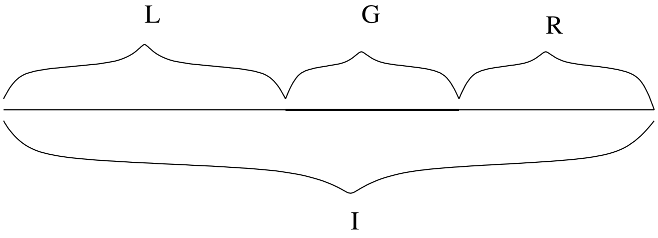

Figure 1

The bridge geometry and the gap geometry of a family of Cantor systems dynamically defined by a family of folding maps depending on a small parameter $\epsilon > 0$ has been studied in [Jia2]. Such families are defined as follows:

Let $\gamma > 1$ be a fixed positive number. The following definition is from [Jia2]. The reader may refer to [Jia2] or [Jia3] for the definition of Schwarzian derivative and its properties. The reader may also refer to Figure 2 for the following definition.

Definition 3. *Let $\mathcal{F} = \{f_\varepsilon : [-1, 1] \to \mathbb{R}\}_{0 \leq \epsilon \leq \epsilon_0}$ be a family of folding mappings. We say $\mathcal{F}$ is asymptotically non-hyperbolic if*

(1) *every $f_\epsilon(x) = h_\epsilon(-|x|^\gamma)$, where $h_\epsilon : [-1, 0] \to [-1, 1+\epsilon]$ is a C^3 orientation-preserving diffeomorphism having non-positive Schwarzian derivative on $[-1, 0]$ such that $h_\epsilon(-1) = -1$ and $h_\epsilon(0) = 1 + \epsilon$,*

(2) *there is a constant $K > 0$ such that $\Big(\log h'_\epsilon(x)\Big)' \leq K$ for all $-1 \leq x \leq 0$ and all $0 \leq \epsilon \leq \epsilon_0$, and*

(3) *there is a constant $\lambda > 1$ such that $f'_\epsilon(-1) \geq \lambda$ for all $0 \leq \epsilon \leq \epsilon_0$.*

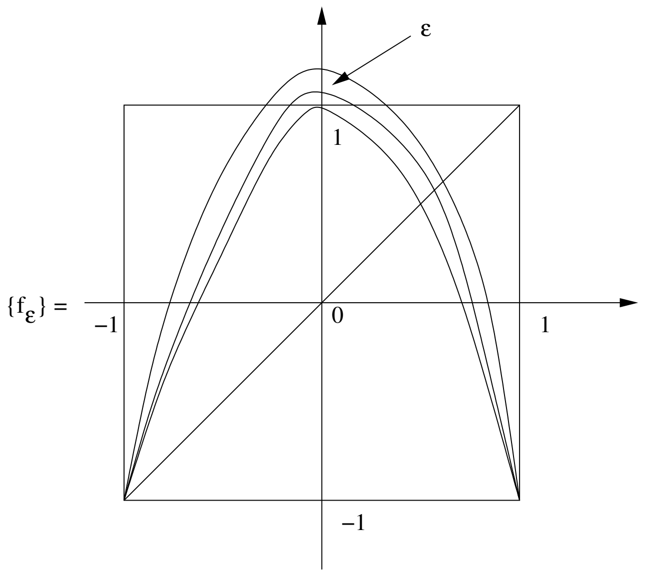

Figure 2

Let $\mathcal{F} = \{f_\varepsilon : [-1, 1] \to \mathbb{R}\}_{0 \leq \epsilon \leq \epsilon_0}$ be an asymptotically non-hyperbolic family. For each $0 < \epsilon \leq \epsilon_0$, let $\mathcal{I}_{n,\epsilon}$ be the set of intervals in (i.e. connected components of) $f_\epsilon^{-n}([-1, 1])$ and let $\mathcal{G}_{n,\epsilon}$ be the set of intervals in (i.e. connected components of) $f_\epsilon^{-n}((1, 1+\epsilon))$. Let

$$\mathcal{I}_\epsilon = \{\mathcal{I}_{n,\epsilon}\}_{n=0}^{\infty} \quad \text{and} \quad \mathcal{G}_\epsilon = \{\mathcal{G}_{n,\epsilon}\}_{n=1}^{\infty}.$$

Then $\mathcal{C}_\epsilon = \{\mathcal{I}_\epsilon, \mathcal{G}_\epsilon\}$ is a Cantor system dynamically defined by f_ϵ. Let

$$\Lambda_\epsilon = \bigcap_{n=0}^{+\infty} \bigcup_{I \in \mathcal{I}_{n,\epsilon}} I$$

be the Cantor set obtained from the Cantor system $\mathcal{C}_\epsilon = \{\mathcal{I}_\epsilon, \mathcal{G}_\epsilon\}$. It is just the maximal invariant set of f_ϵ.

One example of such a family is the maps defined on $[-1, 1]$ by

$$f_\epsilon(x) = 1 + \epsilon - (2+\epsilon)|x|^\gamma$$

where $\gamma > 1$ is fixed and $\epsilon > 0$ is a parameter. In Definition 3, the dynamical system f_ϵ is hyperbolic when $\epsilon > 0$ and ceases to be hyperbolic when $\epsilon = 0$. It was shown in [Jia2] that for Cantor systems defined by f_ϵ, the bridge geometry is uniformly bounded and the gap geometry is regulated by the function $\alpha(\epsilon) = \epsilon^{\frac{1}{\gamma}}$ (see Theorem 2 in §2). These geometry properties are keys for us to prove the following theorem. Let $\dim \Lambda_\epsilon$ denote the Hausdorff dimension of Λ_ϵ. Here is the main theorem in this paper.

Theorem 1 (Main theorem). *There exists constants* $K > 0$ *and* $\epsilon_1 > 0$ *such that*

$$1 - K^{-1}\epsilon^{\frac{1}{\gamma}} \le \dim \Lambda_\epsilon \le 1 - K\epsilon^{\frac{1}{\gamma}}, \quad \forall\, 0 < \epsilon \le \epsilon_1.$$

One can compare $\dim \Lambda_\epsilon$ in the theorem with the Hausdorff dimension $\dim J_c$ of the Julia set J_c of $z \to z^2 + c$. It is known that $\dim J_c$ is a real analytic function of $|c|$ in any hyperbolic component. In particular, Ruelle prove in [Rue] that

$$\dim J_c = 1 + \frac{|c|^2}{4} + o(|c|^2)$$

when c is in the main cardioid of the Mandelbrot set. Thus, when restricted on the real line, $\dim J_c$ is real analytic on $[0, \frac{1}{4}) \cup (\frac{1}{4}, \infty)$. But Bodart and Zinsmeister prove in [BZ] that $\dim J_c$ is left continuous at the point $\frac{1}{4}$, and Douady, Sentenac, and Zinsmeister prove in [DSZ] that $\dim J_c$ is not right continuous at the point $\frac{1}{4}$ and the derivative tends to $+\infty$ from the left at the point $\frac{1}{4}$ like $(\frac{1}{4} - c)^{\dim J_{1/4} - \frac{3}{2}}$. When restricted on the real line, $\dim J_c$ is real analytic on $(-\infty, -2)$. The theorem in this paper shows that $\dim J_c$ is continuous from the left at -2 and the derivative tends to $+\infty$ from the left of -2 like $1/\sqrt{2 - |c|}$. A similar phenomenon occurs in the study of Riemann surfaces. In [PS], Pignataro and Sullivan shows that $\dim \Lambda_\epsilon$ of the limit set Λ_ϵ of the Fuchsian group generated by $z \to -1/z$ and $z \to z + 2 + \epsilon$ differs from 1 by $\sqrt{\epsilon}$ and the derivative at the critical point $\epsilon = 0$ tends to ∞ like $1/\sqrt{\epsilon}$. However, we would like to point out the difference from the cases in [BZ,DSZ,PS] to the case we study: The asymptotic geometry in [BZ,DSZ,PS] is the parabolic type but the asymptotic geometry studied in [Jia1,Jia2] and in this paper is the post-critical finite type.

The paper is organized as follows. In §2, we review the result from [Jia2] about the geometry of Cantor systems generated by an asymptotically non-hyperbolic family of folding mappings (see Theorem 2). The upper bound $1 - K\epsilon^{\frac{1}{\gamma}}$ of the Hausdorff dimension of Λ_ϵ is already known from [Jia1,Jia2]). For the completeness of this paper, we also write the proof in §3 as Lemma 1. The main effort of this paper is to prove the lower bound. We do this in §3 too. The main ingredients in the proof of the lower bound are the asymptotic geometry from [Jia2] (see Theorem 2), an inequality from [KaPe] (see also [Fan]), the Frostman theorem (see [Fal,Mat]), and an idea used to treat a family of linear Cantor sets in [Jia2].

2. ASYMPTOTICALLY NON-HYPERBOLIC FAMILIES

In this paper, we will always use $K > 0$ and $K' > 0$ etc to denote a constant independent of ϵ although it may be different in different places.

Let $\mathcal{BR}_\epsilon$ be the bridge geometry and $\mathcal{GAP}_\epsilon$ be the gap geometry of $\mathcal{C}_\epsilon$.

Theorem 2 [Jia2]. *For each $0 < \epsilon \leq \epsilon_0$, the dynamically defined interval system $\mathcal{C}_\epsilon$ is a Cantor system whose bridge geometry is uniformly bounded:*

$$\Big|\log\big(br(\epsilon)\big)\Big| \leq K \qquad (*)$$

for all $br(\epsilon) \in \mathcal{BR}_\epsilon$ and all $0 < \epsilon \leq \epsilon_0$, and whose gap geometry is regulated by the function $\alpha(\epsilon) = \epsilon^{\frac{1}{\gamma}}$:

$$\Big|\log\Big(\frac{gg(\epsilon)}{\alpha(\epsilon)}\Big)\Big| \leq K \qquad (**)$$

for all $gg(\epsilon) \in \mathcal{GAP}_\epsilon$ and all $0 < \epsilon \leq \epsilon_0$. Moreover, for all $0 < \epsilon \leq \epsilon_0$ and for all integers $n, m, k \geq 1$, if $J \in \mathcal{I}_{n+m+k,\epsilon}$ and $J \subset I \in \mathcal{I}_{n+k,\epsilon}$, then we have $f_\epsilon^k(J) \in \mathcal{I}_{m+n,\epsilon}$ and $f_\epsilon^k(I) \in \mathcal{I}_{n,\epsilon}$ and

$$e^{-K}\frac{|f_\epsilon^k(J)|}{|f_\epsilon^k(I)|} \leq \frac{|J|}{|I|} \leq e^{K}\frac{|f_\epsilon^k(J)|}{|f_\epsilon^k(I)|} \qquad (***).$$

We would like to note that each map f_ϵ in the family is not expanding under the Euclidean metric. Under the hyperbolic metric on $(-2-\epsilon, 2+\epsilon)$, f_ϵ is expanding. But the hyperbolic metric depends on ϵ. So the proof of the above theorem depends on a deep understanding of the Koebe distortion principle [Jia3]. The reader who is interested in the Koebe distortion principle should refer to [Jia3, Chapter 2] for details. The main purpose of this paper is to use the geometry in the above theorem to estimate the Hausdorff dimension of Λ_ϵ from below.

3. THE PROOF OF MAIN THEOREM

The proof is decomposed into several lemmas. We first give the proof of the upper bound obtained in [Jia1,Jia2] as the following lemma.

Lemma 1. *There is a constant $K' > 0$ such that*

$$\dim \Lambda_\epsilon \leq 1 - K'\epsilon^{\frac{1}{\gamma}}.$$

Proof. The family $\mathcal{I}_{n,\epsilon}$ is a natural cover of Λ. Define

$$S_n(\beta) = \sum_{J\in\mathcal{I}_{n,\epsilon}} |J|^\beta, \quad 0 < \beta \le 1.$$

We have only to show $S_n(\beta) = O(1)$ for $\beta > 1 - K\epsilon^{\frac{1}{\gamma}}$.

Intervals in $\mathcal{I}_{n,\epsilon}$ are grouped two by two. Each couple $J, J' \subset \mathcal{I}_{n,\epsilon}$, together with a gap $G \in \mathcal{G}_{n,\epsilon}$, are associated with an interval $I \in \mathcal{I}_{n,\epsilon}$ such that $I = J \cup G \cup J'$. Then

$$S_n(\beta) = \sum_{I=J\cup G\cup J'\in\mathcal{I}_{n-1,\epsilon}} (|J|^\beta + |J'|^\beta).$$

Notice that from $(*)$ and $(**)$ we have

$$|J| + |J'| = |I| - |G| = |I|\Big(1 - \frac{|G|}{|I|}\Big) \le |I|(1 - e^{-2K}\epsilon^{\frac{1}{\gamma}}).$$

We apply the Hölder inequality, $a^\beta + b^\beta \le 2^{1-\beta}(a+b)^\beta$, to get

$$S_n(\beta) \le 2^{1-\beta} \sum_{I\in\mathcal{I}_{n-1,\epsilon}} (|J| + |J'|)^\beta \le 2^{1-\beta}(1 - e^{-2K}\epsilon^{\frac{1}{\gamma}})^\beta S_{n-1}(\beta).$$

Inductively, we get

$$S_n(\beta) \le 2^{(1-\beta)n}\big(1 - e^{-2K}\epsilon^{\frac{1}{\gamma}}\big)^{n\beta}.$$

Let $\beta(\epsilon) > 0$ be the number such that

$$2^{1-\beta(\epsilon)}\big(1 - e^{-2K}\epsilon^{\frac{1}{\gamma}}\big)^{\beta(\epsilon)} = 1.$$

Then $\dim(\Lambda_\epsilon) \le \beta(\epsilon)$. Solving the last equality, we have

$$\dim(\Lambda_\epsilon) \le \beta(\epsilon) = \frac{\ln 2}{\ln 2 - \ln(1 - e^{-2K}\epsilon^{\frac{1}{\gamma}})} \le 1 - K'\epsilon^{\frac{1}{\gamma}}.$$

This completes the proof of Lemma 1. $\square$

Now we show the lower bound for $\dim \Lambda_\epsilon$. Let

$$0 < \kappa = e^{-5K}/2 < 1$$

where K is the constant appearing in (*), (**) and (***). Let $s_n = s_n(\epsilon)$ and $t_n = t_n(\epsilon)$ be two numbers determined by

$$\sum_{I \in \mathcal{I}_{n,\epsilon}} |\kappa I|^{s_n} = 1 \quad \text{and} \quad \sum_{I \in \mathcal{I}_{n,\epsilon}} |I|^{t_n} = 1.$$

It is clear that $0 < s_n \le t_n < 1$.

Lemma 2. *$\lim_{n\to\infty}(t_n - s_n) = 0$.*

Proof. Since $s_n \le t_n$,

$$1 \le \sum_{I \in \mathcal{I}_{n,\epsilon}} |I|^{s_n} \le \sum_{I \in \mathcal{I}_{n,\epsilon}} (\kappa|I|)^{s_n} \kappa^{-s_n} = \kappa^{-s_n} \le \kappa^{-1}.$$

From $(*)$, there is a constant $0 < \tau < 1$ such that $|I| \le \tau^n$ for every $I \in \mathcal{I}_{n,\epsilon}$. We have further that

$$\kappa^{-1} \ge \sum_{I \in \mathcal{I}_{n,\epsilon}} |I|^{s_n} = \sum_{I \in \mathcal{I}_{n,\epsilon}} |I|^{t_n} |I|^{-(t_n - s_n)} \ge \tau^{-n(t_n - s_n)}.$$

This implies that

$$0 \le t_n - s_n \le \frac{\ln \kappa^{-1}}{n \ln \tau^{-1}}.$$

This proves Lemma 2. $\square$

The following lemma used in [KaPe] (see also [Fan]) is also useful for us.

Lemma 3. *$a^\beta + \beta b^\beta \ge (a+b)^\beta$, for all $a \ge b \ge 0$ and for all $0 < \beta \le 1$.*

Proof. The inequality is equivalent to $\beta x^\beta \ge (1+x)^\beta - 1$ for all $0 \le x \le 1$. But the later one is a direct consequence of the mean value theorem applied to $(1+x)^\beta$. $\square$

Lemma 4. *$\dim(\Lambda_\epsilon) \ge s_n$.*

Proof. Let $g = f_\epsilon^n$. We let n be fixed in this lemma. Consider the sub-Cantor systems $(\{\mathcal{I}_{mn,\epsilon}\}_{m\ge1}, \{\mathcal{G}_{mn,\epsilon}\}_{m\ge1})$. Define a measure μ on Λ_ϵ as follows:

Suppose $J_{mn} \in \mathcal{I}_{mn,\epsilon}$. Then $g^i(J_{mn}) \in \mathcal{I}_{(m-i)n,\epsilon}$ for $0 \leq i \leq m-1$. Let $I_i \in \mathcal{I}_{n,\epsilon}$ be the unique interval containing $g^i(J_{mn})$ (see Figure 3). Then define

$$\mu(\Lambda_\epsilon \cap J_{mn}) = (\kappa|I_0)^{s_n}(\kappa|I_1|)^{s_n} \cdots (\kappa|I_{m-1}|)^{s_n}.$$

Note that if $J_{(m-1)n} \in \mathcal{I}_{(m-1)n,\epsilon}$ is the interval containing J_{mn}, then

$$\mu(\Lambda_\epsilon \cap J_{(m-1)n}) = (\kappa|I_0)^{s_n}(\kappa|I_1|)^{s_n} \cdots (\kappa|I_{m-2}|)^{s_n}.$$

Therefore,

$$\mu(\Lambda_\epsilon \cap J_{mn}) = \mu(\Lambda_\epsilon \cap J_{(m-1)n})(\kappa|I_{m-1}|)^{s_n}$$

and consequently

$$\mu(\Lambda_\epsilon \cap J_{(m-1)n}) = \sum_{J_{mn} \subset J_{(m-1)n}} \mu(\Lambda_\epsilon \cap J_{mn}).$$

So, according to the Kolmogorov extension theorem, μ is a well defined Borel probability measure concentrated on Λ_ϵ.

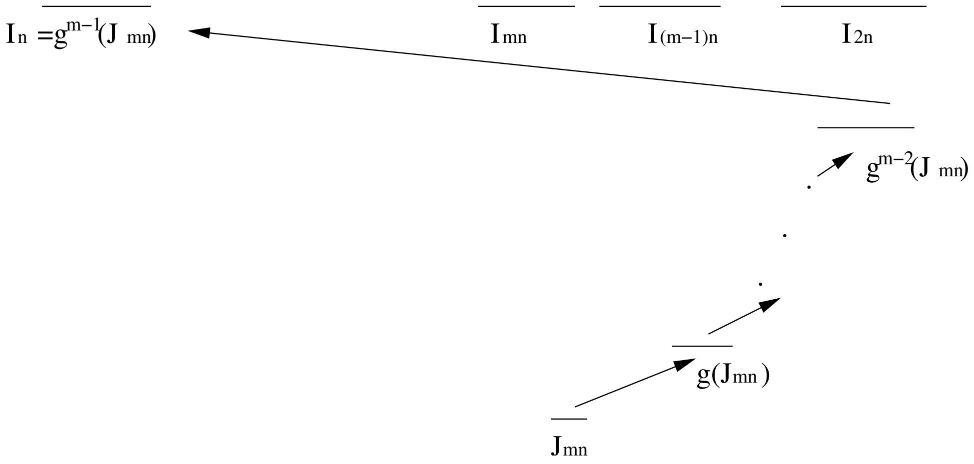

Figure 3

We now estimate $\mu(B_r(x))$, where $B_r(x)$ is the ball centered x of radius r. Let

$$\delta = \min \left\{ |G| \mid G \in \bigcup_{1 \leq k \leq n} \mathcal{G}_{k,\epsilon} \right\}$$

be the minimal gap-length of gaps in $\cup_{1\leq k\leq n}\mathcal{G}_{k,\epsilon}$. Assume $x\in\Lambda_\epsilon$. There is a sequence of nested intervals

$$x\in\cdots\subset J_{mn}\subset J_{(m-1)n}\subset\cdots J_{2n}\subset J_n,\quad \text{where } J_{mn}\in\mathcal{I}_{mn,\epsilon}.$$

Assume $0<r<d$. Let m be the unique integer such that

$$\delta^{s_n}\mu(J_{mn})\leq r^{s_n}<\delta^{s_n}\mu(J_{n(m-1)}).$$

Adjacent to J_{nm} there are two gaps. One is a mn-level gap $G\in\mathcal{G}_{mn,\epsilon}$ and the other is a lower level gap G'. Any other interval $J'\in\mathcal{I}_{mn,\epsilon}$ different from J_{nm} is separated from J_{nm} by a gap of length at least $\min\{|G'|,|G|\}$. We claim that

$$\min\{|G'|,|G|\}\geq\delta\big(\mu(J_{n(m-1)})\big)^{1/s_n}.$$

Suppose the claim is true. Then $B_r(x)$ intersects with J_{nm} but none of the others in $\mathcal{I}_{mn,\epsilon}$. Thus

$$\mu(\Lambda_\epsilon\cap B_r(x))\leq\mu(J_{nm})\leq\delta^{-s_n}r^{s_n}.$$

So by the mass distribution principle (see [Fal]), we get $\dim(\Lambda_\epsilon)\geq s_n$.

We now prove the claim. First from the gap geometry $(**)$, we have

$$|G|\geq e^{-K}\epsilon^{\frac{1}{\gamma}}|J_{nm}|.$$

Write

$$|J_{nm}|=\frac{|J_{nm)}|}{|J_{n(m-1)}|}\cdots\frac{|J_{n(m-i+1)}|}{|J_{n(m-i))}|}\cdots\frac{|J_{2n}|}{|J_n|}\cdot\frac{|J_n|}{|G_n|}\cdot|G_n|,$$

where G_n is the gap in $\mathcal{G}_{n,\epsilon}$ adjacent to J_n (see Figure 4). We have, again from $(**)$,

$$\frac{|J_n|}{|G_n|}\geq\frac{e^{-K}}{\epsilon^{\frac{1}{\gamma}}}.$$

We also have $|G_n| \ge \delta$ by the definition of δ. Using the first inequality in $(***)$ (refer to Figure 4), we get

$$\frac{|J_{n(m-(i+1))}|}{|J_{n(m-i)}|} \ge e^{-K}\frac{|g^{m-i}(J_{n(m-(i+1))}|}{|g^{m-i}(J_{n(m-i)})|} = e^{-K}\frac{|I_i|}{|[-1,1]|} = e^{-K}\frac{|I_i|}{2}.$$

Thus we have

$$|G| \ge e^{-K}\epsilon^{\frac{1}{\gamma}} \cdot \prod_{k=1}^{m-1}\frac{e^{-K}|I_k|}{2} \cdot \frac{e^{-K}}{\epsilon^{\frac{1}{\gamma}}} \cdot \delta$$

$$= e^{-2K} \cdot \prod_{k=1}^{m-1}\frac{e^{-K}|I_k|}{2} \cdot \delta.$$

On the other hand, assume $G' \in \mathcal{G}_{k,\epsilon}$ for some $1 \le k < mn$. Let I' be the interval in $\mathcal{I}_{k,\epsilon}$ adjacent to G' but containing J_{mn}. Then

$$|G'| \ge e^{-K}\epsilon^{\frac{1}{\gamma}}|I'| = e^{-K}\epsilon^{\frac{1}{\gamma}}\frac{|I'|}{|J_{mn}|}\frac{|J_{mn}|}{|G|}|G| \ge e^{-K}\epsilon^{\frac{1}{\gamma}} \cdot 1 \cdot \frac{e^{-K}}{\epsilon^{\frac{1}{\gamma}}}|G| = e^{-2K}|G|.$$

Thus

$$\min\{|G'|,|G|\} \ge e^{-4K} \cdot \prod_{k=1}^{m-1}\frac{e^{-K}|I_k|}{2} \cdot \delta$$

$$\ge (\kappa|I_1)(\kappa|I_2)\cdots(\kappa|I_{m-1}|)\delta = \delta\big(\mu(J_{(m-1)n})\big)^{1/s_n}.$$

This proves the claim. ☐

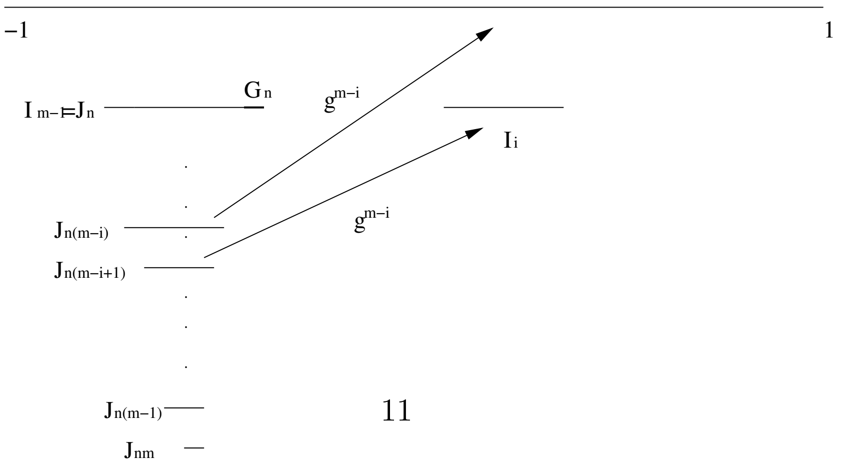

Figure 4

Lemma 5. $\limsup_{n\to\infty} t_n \geq 1 - K^{-1}\epsilon^{\frac{1}{\gamma}}$.

Proof. The upper bound in our main theorem will be a consequence of Lemma 2, Lemma 4 and Lemma 5. We are going to estimate from below the quantity

$$S_n(\beta) = \sum_{J\in\mathcal{I}_{n,\epsilon}} |J|^\beta.$$

As in our lower bound estimation, for each $I \in \mathcal{I}_{n-1,\epsilon}$, let J and J' be the two intervals in $\mathcal{I}_{n,\epsilon}$ such that $I = J \cup G \cup J' \in \mathcal{I}_{n-1,\epsilon}$ where $G \in \mathcal{G}_{n,\epsilon}$ is the n-level gap contained in I. Assume $|J'| \geq |J|$ without loss of generality. Then by Lemma 3, we have

$$\begin{aligned} |J'|^\beta + |J|^\beta &= (1-\beta)|J|^\alpha + \big(|J'|^\beta + \beta|J|^\beta\big) \\ &\geq (1-\beta)\Big(\frac{|J|}{|I|}\Big)^\beta |I|^\beta + \big(|J'| + |J|\big)^\beta \\ &= \Big[(1-\beta)\Big(\frac{|J|}{|I|}\Big)^\beta + \Big(1 - \frac{|G|}{|I|}\Big)^\beta\Big]|I|^\beta. \end{aligned}$$

Notice that $(*)$ and $(**)$ imply

$$\Big(\frac{|J|}{|I|}\Big)^\beta \geq e^{-\beta K} \geq e^{-K} \quad \text{and} \quad 1 - \frac{|G|}{|I|} \geq 1 - e^K\epsilon^{\frac{1}{\gamma}}.$$

So, let $A = e^K > 1$. Then

$$|J'|^\beta + |J|^\beta \geq f(\beta)|I|^\beta,$$

where

$$f(\beta) = (1-\beta)A^{-1} + (1 - A\epsilon^{\frac{1}{\gamma}})^\beta.$$

Thus we have

$$S_n(\beta) = \sum_{I\in\mathcal{I}_{n-1,\epsilon}} (|J|^\beta + |J'|^\beta) \geq f(\beta)S_{n-1}(\beta).$$

Inductively,

$$S_n(\beta) \geq (f(\beta))^n.$$

Since $f(0) > 1$ and $f(1) < 1$ and $f'(\beta) < 0$ for $0 \leq \beta \leq 1$ and small $\epsilon > 0$, there is an $\epsilon_1 > 0$ such that for every $0 \leq \epsilon \leq \epsilon_1$, there is a unique solution $0 < \beta(\epsilon) \leq 1$ for the equation $f(\beta) = 1$, i.e.

$$(1 - \beta(\epsilon))A^{-1} + (1 - A\epsilon^{\frac{1}{\gamma}})^{\beta(\epsilon)} = 1.$$

Then $t_n \geq \beta(\epsilon)$. Now we only need to estimate $\beta(\epsilon)$.

From the implicit function theorem, $\beta(\epsilon)$ is continuous on $[0, \epsilon_1]$ and C^1 on $(0, \epsilon_1]$. Differentiate on the both sides of the last equality to get

$$\beta'(\epsilon) = -\frac{A\beta(\epsilon)(1 - A\epsilon^{\frac{1}{\gamma}})^{\beta(\epsilon)-1}}{\gamma\big(A^{-1} - \big(1 - A\epsilon^{\frac{1}{\gamma}}\big)^{\beta(\epsilon)} \ln(1 - A\epsilon^{\frac{1}{\gamma}})\big)} \cdot \epsilon^{\frac{1}{\gamma}-1}.$$

So there is a constant $K > 0$ such that

$$\beta'(\epsilon) \geq -\frac{1}{K\gamma}\epsilon^{\frac{1}{\gamma}-1}.$$

Integrate on both sides, and notice that $\beta(0) = 1$, we have

$$\beta(\epsilon) \geq 1 - K^{-1}\epsilon^{\frac{1}{\gamma}}.$$

This completes the proof. ☐

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