Commun. Math. Phys. 223, 143 – 159 (2001)

Communications in
Mathematical Physics

On Ruelle–Perron–Frobenius Operators. II. Convergence Speeds

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Received: 31 May 2000 / Accepted: 1 June 2001

Abstract: We study Ruelle operators on expanding and mixing dynamical systems with potential function satisfying the Dini condition. We give an estimate for the convergence speed of the iterates of a Ruelle operator. Our proof avoids Markov partitions. This is the second part of our research on Ruelle operators.

1. Introduction

Let X be a compact metric space with metric d and $f : X \to X$ be a continuous map. The couple (X, f) is called a dynamical system. Let $\psi : X \to \mathbb{R}_*^+$ be a strictly positive continuous function, called a *potential*. The Ruelle–Perron–Frobenius operator $\mathcal{L} = \mathcal{L}_{f,\psi}$, simply called the Ruelle operator, is defined as

$$\mathcal{L}\phi(x) = \sum_{y \in f^{-1}(x)} \psi(y)\phi(y)$$

for ϕ in a suitable space of functions on X. The Ruelle operator is an important tool in the study of dynamical systems. The famous Ruelle theorem deals with spectral properties of $\mathcal{L}$ and then implies the convergence of the powers of $\mathcal{L}$. Under the setting of an expanding and mixing dynamical system with a Dini potential, the Ruelle theorem is proved in the first part of our study [FJ]. (See also [Ru1, Ru2, Bo, Wa] for different proofs under different settings.)

In this paper, we present our results on the convergence speed of the powers of $\mathcal{L}$. Our result is new in the general setting that we consider here. Our method may also work in the setting considered in [Wa], where no convergence speed was studied.

Recall that a dynamical system f on X is said to be *locally expanding* if there are constants $\lambda > 1$ and $b > 0$ such that

$$d(f(x), f(y)) \geq \lambda d(x, y), \quad x, y \in X, \ \ d(x, y) \leq b.$$

⋆ Supported in part by NSF grants and PSC-CUNY awards.

We call (λ, b) a *primary expanding parameter*. It is said to be *mixing* if for any non-empty open set U of X, there is an integer $n > 0$ such that $f^n(U) = X$. For any $n \geq 0$, we define a new metric d_n on X, called the *n-Bowen metric*, as

$$d_n(x, y) = \max_{0 \leq j \leq n} d(f^j(x), f^j(y)).$$

The *n-Bowen ball* centered at $x \in X$ of radius $r > 0$ is denoted by $B_n(x, r)$. The 0-Bowen metric is just the original metric d on X.

Let $\mathcal{C} = \mathcal{C}(X, \mathbb{R})$ be the space of all continuous functions $\phi : X \to \mathbb{R}$ with the supremum norm

$$||\phi||_\infty = \max_{x \in X} |\phi(x)|.$$

For a right continuous and increasing function $\omega : \mathbb{R}^+ \to \mathbb{R}^+$ with $\omega(0) = 0$ (called *modulus of continuity*), we define $\mathcal{H}^\omega = \mathcal{H}^\omega(X, \mathbb{R})$ to be the space of all functions $\phi \in \mathcal{C}$ satisfying

$$[\phi]_\omega = \sup_{x, y \in X, 0 < d(x,y) \leq a} \frac{|\phi(x) - \phi(y)|}{\omega(d(x, y))} < \infty.$$

(The number $a > 0$ will be chosen and fixed later.) A modulus of continuity $\omega(t)$ is said to satisfy the *Dini condition* if

$$\int_0^1 \frac{\omega(t)}{t} dt < \infty.$$

For such a Dini function ω, define

$$\tilde{\omega}(t) = \sum_{n=1}^{\infty} \omega(\lambda^{-n} t).$$

It is easy that $\tilde{\omega}$ is also a modulus of continuity.

Let $\mathcal{M}$ be the dual space of $\mathcal{C}$ and let $\mathcal{L}^* : \mathcal{M} \to \mathcal{M}$ be the dual operator of $\mathcal{L} : \mathcal{C} \to \mathcal{C}$. For any measure $\nu \in \mathcal{M}$ and any function $\phi \in \mathcal{C}$, we use $\langle \nu, \phi \rangle$ to denote the integral of ϕ with respect to ν.

Let us recall the Ruelle theorem that we proved in [FJ].

Theorem 1 (Ruelle Theorem**).** *Suppose that ω is a Dini modulus of continuity and $\psi \in H^\omega$. We have the following statements:*

1. *There exists a strictly positive number ρ and a strictly positive function $h \in \mathcal{H}^{\tilde{\omega}}$ such that $\mathcal{L}h = \rho h$.*
2. *There exists a unique probability measure $\nu = \nu_\psi \in \mathcal{M}$ such that $\mathcal{L}^* \nu = \rho \nu$.*
3. *For sufficiently small $r > 0$, there is a constant $C = C(r) > 0$ such that*

$$C^{-1} \leq \frac{\nu\big(B_n(x, r)\big)}{\rho^{-n} G_n(x)} \leq C \quad \textit{(Gibbs property)}$$

holds for all $x \in X$ and $n \geq 1$, where $G_n(x) = \prod_{j=0}^{n-1} \psi(f^j x)$.

4. *Take h in (1) such that $\langle \nu, h \rangle = 1$. Then for any $\phi \in \mathcal{C}$,*

$$\rho^{-n} \mathcal{L}^n \phi \to \langle \nu, \phi \rangle h \quad \textit{as} \quad n \to \infty.$$

Notice that the function h belongs to $\mathcal{H}^{\tilde{\omega}}$ but not to $\mathcal{H}^{\omega}$, in general.

Our concern in this paper is the convergence speed of $\rho^{-n}\mathcal{L}^n\phi$. Such speeds will provide us with good knowledge on the statistical properties of the dynamical system. We shall see that the convergence speed depends on the regularities of both ψ and ϕ. For any function ϕ, denote by $\Omega_\phi(t)$ its modulus of continuity defined by $\sup_{d(x,y)\le t}|\phi(x)-\phi(y)|$. Our main result in this paper is the following.

Theorem 2. *Make the same assumptions as in Theorem 1. Take an eigenfunction h (associated to the eigenvalue ρ) such that $\langle \nu, h\rangle = 1$. Then for any ϵ with $0 < c_2\epsilon \le a$, $c_2 = 2\lambda/(\lambda-1)$, there exist constants $0<\gamma<1$, $p \ge 0$, $C > 0$ such that for any $n \ge 1$, any $\phi \in \mathcal{C}$, any integer partition of $[1, n]$, $1 \le n_0 < n_1 < \cdots < n_{\ell-1} < n_\ell \le n$, satisfying $n_j - n_{j-1} > p$ for $1 \le j < \ell$ (let $n_{-1} = 0$), we have*

$$\|\rho^{-n}\mathcal{L}^n\phi - \langle \nu, \phi\rangle h\|_\infty$$
$$\le C\Big(\Omega_\phi(c_2\epsilon\lambda^{-n_0}) + \|\phi\|_\infty \sum_{j=0}^{\ell} \tilde{\omega}(\lambda^p c_2\epsilon\lambda^{-(n_j-n_{j-1})}) + \|\phi\|_\infty \gamma^\ell\Big).$$

Our result in the general setting unifies to some extent the existing ones (see [FP]). Our method is completely different and seems simple. Markov partitions are not needed, unlike what one could expect. That is one reason for the simplicity of the method. In the place of the Markov partition, we need a non Markovian partition which is very easy to construct and may be adapted to the setting studied in [Wa].

The article is organized as follows. In Sect. 2, we will recall some properties of an expanding and mixing dynamical system and construct non Markovian partitions. In Sect. 3, we will prove our main result (Theorem 2, also Theorems 3 and 4). In Sect. 4, we will give some examples providing different kind of convergence speeds (polynomial, superpolynomial, subexponential, etc). In Sect. 5, we will apply the main result to get decays of correlation and the central limit theorem.

2. Construction of Non-Markovian Partitions

For a locally expanding dynamical system (X, f) with expanding primary parameter (λ, b), the restriction $f : B(x, b') \to f(B(x, b'))$ is homeomorphic for any $x \in X$ and $0 < b' \le b$. Moreover, there is an integer $m_0 > 0$ such that $\#(f^{-1}(x)) \le m_0$ for any $x \in X$ and for any $x \in X$ and any $0 < r \le b$, we have

$$B_k(x, r) \subseteq B_{k-1}(x, \lambda^{-1}r) \quad (k \ge 1).$$

Some further properties listed below are proved in [FJ].

Proposition 1. *Suppose f is a locally expanding and mixing dynamical system with a primary expanding parameter (λ, b).*

1. *There is a constant $0 < a \le b$ such that for any $x \in X$ with $f^{-1}(x) = \{x_1, \cdots, x_n\}$, there are local inverses $g_1, \cdots, g_n$ of f defined on $B(x, a)$ such that $g_j(x) = x_j$ and $g_j(B(x, a))$ ($1 \le j \le n$) are pairwise disjoint.*
2. *Let $a > 0$ be a constant in (1). We have $\#(f^{-1}(x)) = \#(f^{-1}(y))$ if $d(x, y) \le a$. Furthermore, we can arrange $f^{-1}(x) = \{x_1, \cdots, x_n\}$ and $f^{-1}(y) = \{y_1, \cdots, y_n\}$ so that*

$$d(x_j, y_j) \le \frac{d(x, y)}{\lambda} \quad (1 \le j \le n).$$

3. *Let* $a > 0$ *be a constant in (1). If* $0 < r \leq a$*, the map*

$$f^n : B_n(x, r) \to B(f^n(x), r)$$

is a homeomorphism.

4. *Let* $a > 0$ *be a constant in (1). Then for any* $0 < r \leq a$*, there is an integer* $p = p(r) \geq 1$ *such that* $f^p(B(x, r)) = X$ *for any* $x \in X$*. Moreover, for any* $x, y \in X$*,*

$$1 \leq \#\big(f^{-(n+p)}(y) \cap B_n(x, r)\big) \leq m_0^p.$$

Let a be a constant in (1). We call the pair (λ, a) an *expanding parameter* for f. Henceforth we suppose f is a locally expanding and mixing dynamical system with a fixed expanding parameter (λ, a).

Now we are going to construct a sequence of partitions of X when $\lambda > 3$. Denote

$$c_1 = \frac{\lambda - 3}{\lambda - 1} \quad \text{and} \quad c_2 = \frac{2\lambda}{\lambda - 1}.$$

Let ϵ be a real number satisfying that $0 < 2\epsilon \leq a$. Let $\{x_1, \cdots, x_m\}$ be a 2ϵ-net in (X, d), that is to say, the balls $\{B(x_j, \epsilon)\}_{1 \leq j \leq m}$ are disjoint and the balls $\{B(x_j, 2\epsilon)\}_{1 \leq j \leq m}$ form a cover of X. Define

$$A_1 = B(x_1, 2\epsilon) \setminus \big(B(x_2, \epsilon) \cup \cdots \cup B(x_m, \epsilon)\big),$$
$$A_j = B(x_j, 2\epsilon) \setminus (A_1 \cup \cdots \cup A_{j-1}) \quad (2 \leq j \leq m).$$

Thus we get a partition $\mathcal{P}_0 = \{A_j\}$ of X such that

$$B(x_j, \epsilon) \subseteq A_j \subseteq B(x_j, 2\epsilon) \quad (1 \leq j \leq m).$$

For any $n \geq 1$ and $1 \leq j \leq m$, the inverse under f^n of every A_j is composed of disjoint sets (called components), each of which contains a d_n-ball of radius ϵ and is contained in a d_n-ball of radius 2ϵ (Proposition 1 (3)). More precisely, for each component A, $f^n : A \to A_j$ is homeomorphic and

$$B_n(c_A, \epsilon) \subseteq A \subseteq B_n(c_A, 2\epsilon),$$

where $c_A \in A$ and $f^n(c_A) = x_j$. We call c_A the *center of* A. The set of all such components A form a partition, which we denote by $\mathcal{P}_n$. It is worthy to note that if $n > k \geq 1$ and if $A \in \mathcal{P}_n$, we have $f^k A \in \mathcal{P}_{n-k}$. However $\mathcal{P}_n$ is not necessarily a refinement of $\mathcal{P}_k$. In the following, we will modify $\{\mathcal{P}_k\}_{k=0}^n$ to get a new (finite) sequence of partitions $\{\mathcal{Q}_k\}_{k=0}^n$ such that $\mathcal{Q}_{k+1}$ is a refinement of $\mathcal{Q}_k$.

Proposition 2. *Suppose* $\lambda > 3$*. For any* $n \geq 1$ *and* ϵ *such that* $0 < c_2\epsilon \leq a$*, there are partitions* $\mathcal{Q}_k$ $(0 \leq k \leq n)$ *such that*

1. $\mathcal{Q}_{k+1}$ *is a refinement of* $\mathcal{Q}_k$ $(0 \leq k < n)$*.*
2. *Any element in* $\mathcal{Q}_k$ *contains a* d_k*-ball of radius* $c_1\epsilon$ *and is contained in a* d_k*-ball of radius* $c_2\epsilon$*.*

Proof. We construct $\mathcal{Q}_k$ $(0 \le k \le n)$ by induction on k (decreasing from n to 0). First take $\mathcal{Q}_n = \mathcal{P}_n$. For $A \in \mathcal{P}_{n-1}$, let

$$\tilde{A} = \cup_{D \in \mathcal{Q}_n :\, c_D \in A} D,$$

where c_D is the center of $D \in \mathcal{Q}_n = \mathcal{P}_n$. We claim that

$$B_{n-1}\big(c_A, \epsilon(1-2\lambda^{-1})\big) \subseteq \tilde{A} \subseteq B_{n-1}\big(c_A, 2\epsilon(1+\lambda^{-1})\big).$$

In fact, suppose that the center c_D of $D \in \mathcal{Q}_n$ is outside A. Since A contains the d_{n-1}-ball $B_{n-1}\big(c_A, \epsilon\big)$ of radius ϵ centered at c_A, $d_{n-1}(c_A, c_D) \ge \epsilon$. This implies that for $z \in D \subseteq B_n(c_D, 2\epsilon) \subset B_{n-1}(c_D, 2\epsilon/\lambda)$ we have

$$d_{n-1}(c_A, z) \ge d_{n-1}(c_A, c_D) - d_{n-1}(c_D, z) \ge \epsilon(1-2\lambda^{-1}).$$

Thus we have proved the first inclusion. On the other hand, suppose that the center c_D of $D \in \mathcal{Q}_n$ is inside A. Since A is contained in a d_{n-1}-ball $B_{n-1}\big(c_A, 2\epsilon\big)$ of radius 2ϵ centered at c_A, $d_{n-1}(c_D, c_A) \le 2\epsilon$. This implies that for $z \in D \subseteq B_n(c_D, 2\epsilon) \subset B_{n-1}(c_D, 2\epsilon/\lambda)$, we have

$$d_{n-1}(z, c_A) \le d_{n-1}(z, c_D) + d_{n-1}(c_D, , c_A) \le 2\epsilon\lambda^{-1} + 2\epsilon < 2\epsilon(1+\lambda^{-1}).$$

Thus the second inclusion is proved. All these $\tilde{A}$ form $\mathcal{Q}_{n-1}$. Again we call $c_{\tilde{A}} = c_A$ the center of $\tilde{A}$ in $\mathcal{Q}_{n-1}$. In case there is no confusion, we will still use A (without tilde) to mean an element in $\mathcal{Q}_{n-1}$. Let

$$s_1 = \epsilon(1-2\lambda^{-1}), \quad t_1 = 2\epsilon(1+\lambda^{-1}).$$

Suppose we have constructed $\mathcal{Q}_{n-(k-1)}$ $(2 \le k \le n)$ such that for any $D \in \mathcal{Q}_{n-(k-1)}$ we have

$$B_{n-(k-1)}(c_D, s_{k-1}) \subseteq D \subseteq B_{n-(k-1)}(c_D, t_{k-1}),$$

where c_D is the center of D.

Now for any $A \in \mathcal{P}_{n-k}$, define an element $\tilde{A}$ of $\mathcal{Q}_{n-k}$ as follows:

$$\tilde{A} = \cup_{D \in \mathcal{Q}_{n-k+1} :\, c_D \in A} D.$$

Let c_A be the center of A. We claim that

$$B_{n-k}(c_A, \epsilon - t_{k-1}) \subseteq \tilde{A} \subseteq B_{n-k}\big(c_A, 2\epsilon + t_{k-1}\lambda^{-1}\big).$$

In fact, A in $\mathcal{P}_{n-k}$ contains the d_{n-k}-ball $B_{n-k}(c_A, \epsilon)$ and is contained in the d_{n-k}-ball $B_{n-k}(c_A, 2\epsilon)$. Suppose D is in $\mathcal{Q}_{n-(k-1)}$ whose center c_D is outside A. Then $d_{n-k}(c_A, c_D) \ge \epsilon$. Hence, for any $z \in D \subseteq B_{n-(k-1)}(c_D, t_{k-1}) \subseteq B_{n-k}(c_D, t_{k-1}\lambda^{-1})$, we have

$$d_{n-k}(c_A, z) \ge d_{n-k}(c_A, c_D) - d_{n-k}(c_D, z) \ge \epsilon - t_{k-1}\lambda^{-1} > \epsilon - t_{k-1}.$$

This proves the first inclusion in the claim. On the other hand, for every D in $\mathcal{Q}_{n-(k-1)}$ whose center c_D is in A, we have that $d_{n-k}(c_A, c_D) \le 2\epsilon$ and that for any $z \in D \subseteq B_{n-(k-1)}(c_D, t_{k-1}) \subseteq B_{n-k}(c_D, t_{k-1}\lambda^{-1})$, $d_{n-k}(z, c_D) \le t_{k-1}\lambda^{-1}$. Thus,

$$d_{n-k}(z, c_A) \le d_{n-k}(z, c_D) + d_{n-k}(c_D, c_A) \le 2\epsilon + t_{k-1}\lambda^{-1}.$$

This is the second inclusion in the claim. Now let

$$s_k = \epsilon - t_{k-1}\lambda^{-1} \quad \text{and} \quad t_k = 2\epsilon + t_{k-1}\lambda^{-1}.$$

For any $\tilde{A}$ in $\mathcal{Q}_{n-k}$,

$$B_{n-k}(c_{\tilde{A}}, s_k) \subseteq \tilde{A} \subseteq B_{n-k}\big(c_{\tilde{A}}, t_k\big),$$

where $c_{\tilde{A}} = c_A$ is the center of $\tilde{A}$. An easy calculation shows that

$$t_k = 2\epsilon \frac{\lambda - \lambda^{-k}}{\lambda - 1} \quad \text{and} \quad s_k = \epsilon\Big(1 - \frac{2(1 - \lambda^{-(k+1)})}{\lambda - 1}\Big).$$

We see that $t_k \le c_2\epsilon$ and that for $\lambda > 3$, $s_k \ge c_1\epsilon > 0$. So we have completed the proof. □

3. Convergence Speeds of Ruelle Operators

We give here a proof of Theorem 2.

Let $s = \min_{x\in X} \psi(x)$. Let $K_0 = [\psi]_\omega/s$. For any $x, y \in X$, let $x_i = f^i(x)$ and $y_i = f^i(x)$ for $i \ge 0$. The following distortion property is easy to obtain by using the fact $d(x_i, y_i) \le \lambda^{n-i} d(x_n, y_n)$ for $0 \le i < n$ (a detailed proof is given in [FJ]).

Lemma 1 (Naive Distortion**).** *For any $x, y \in X$ with $d_n(x, y) \le a$,*

$$\left|\log\left(\frac{G_n(x)}{G_n(y)}\right)\right| \le K_0\tilde{\omega}(d(x_n, y_n)),$$

where $G_n(x) = \prod_{j=0}^{n-1} \psi(f^j x)$.

Given $\phi \in \mathcal{C}$. Let $\tilde{\phi} = \phi - \langle \nu, \phi\rangle h$. Then we have $\int \tilde{\phi} d\nu = 0$. And moreover,

$$\rho^{-n}\mathcal{L}^n\tilde{\phi} = \rho^{-n}\mathcal{L}^n\phi - \langle \nu, \phi\rangle h$$

and $\|\tilde{\phi}\|_\infty \le (1 + \|h\|_\infty)\|\phi\|_\infty$. Therefore, Theorem 2 is a consequence of the following theorem.

Theorem 3. *Make the same assumptions as Theorem 2. Then for any ϵ such that $0 < c_2\epsilon \le a$, there exist constants $0 < \gamma < 1$, $p \ge 0$, $C > 0$ such that for any $n \ge 1$, any $\phi \in \mathcal{C}$ such that $\langle \nu, \phi\rangle = 0$, any integer partition of $[1, n]$, $1 \le n_0 < n_1 < \cdots < n_{\ell-1} < n_\ell \le n$, satisfying $n_j - n_{j-1} > p$ for $1 \le j < \ell$ (let $n_{-1} = 0$), we have*

$$\|\rho^{-n}\mathcal{L}^n\phi\|_\infty \le C\Bigg(\Omega_\phi(c_2\epsilon\lambda^{-n_0}) + \|\phi\|_\infty \sum_{j=0}^{\ell} \tilde{\omega}(\lambda^p c_2\epsilon\lambda^{-(n_j - n_{j-1})}) + \|\phi\|_\infty\gamma^\ell\Bigg).$$

Instead of working with the operator $\mathcal{L}$, we shall work with its normalization $\tilde{\mathcal{L}}$, which is defined as follows. Let

$$\tilde{\psi} = \psi \frac{h}{\rho\, h \circ f}.$$

Define

$$\tilde{\mathcal{L}}\phi(x) = \sum_{y\in f^{-1}(x)} \tilde{\psi}(y)\phi(y).$$

The important feature for $\tilde{\mathcal{L}}$ is that $\tilde{\mathcal{L}}1 = 1$. Denote

$$\tilde{G}_n(x) = \prod_{i=0}^{n-1} \tilde{\psi}(f^i(x)) = G_n(x)\frac{h(x)}{\rho^n\, h\circ f^n(x)}.$$

Then we have the expression

$$\tilde{\mathcal{L}}^n\phi(x) = \sum_{y\in f^{-1}(x)} \tilde{G}_n(y)\phi(y).$$

The following lemma is an easy consequence of Lemma 1.

Lemma 2. *Let $K_1 = K_0 + 2[h]_{\tilde{\omega}}/\min h$. For any $x, y \in X$ with $d_n(x, y) \le a$,*

$$\left|\log\left(\frac{\tilde{G}_n(x)}{\tilde{G}_n(y)}\right)\right| \le K_1\tilde{\omega}(d(x_n, y_n)).$$

Moreover, if $K = K_1 e^{K_1\tilde{\omega}(a)}$, we have

$$\left|\frac{\tilde{G}_n(x)}{\tilde{G}_n(y)} - 1\right| \le K\tilde{\omega}(d(x_n, y_n)).$$

Remark that for $0 < \delta \le a$ and $1 \le k \le m$, by Lemma 2 we have

$$\sup\left\{\left|\frac{\tilde{G}_k(x)}{\tilde{G}_k(y)} - 1\right|;\quad d_m(x, y) \le \delta\right\} \le K\tilde{\omega}(\lambda^{-(m-k)}\delta).$$

Let ν be the measure in Theorem 1 (2) and take h in Theorem 1 (1) such that $\langle\nu, \phi\rangle = 1$. Let $\mu = h\nu$ (the Gibbs measure). We will show that Theorem 3 follows from the following theorem.

Theorem 4. *Make the same assumptions as in Theorem 2. Then for any ϵ with $0 < c_2\epsilon \le a$, there exist constants $0 < \gamma < 1$, $p \ge 0$, $K > 0$ such that for any $n \ge 1$, any $\phi \in \mathcal{C}$ such that $\langle\mu, \phi\rangle = 0$, any integer partition of $[1, n]$, $1 \le n_0 < n_1 < \cdots < n_{\ell-1} < n_\ell \le n$, satisfying $n_j - n_{j-1} > p$ for $1 \le j < \ell$, we have that*

$$\|\tilde{\mathcal{L}}^n\phi\|_\infty \le \Omega_\phi(c_2\epsilon\lambda^{-n_0}) + K\|\phi\|_\infty \sum_{j=1}^{\ell} \tilde{\omega}(\lambda^{-(n_j-n_{j-1})}c_2\epsilon\lambda^p) + \|\phi\|_\infty\gamma^\ell.$$

Notice that the sum in the inequality of Theorem 3 is taken over $0 \le j \le \ell$, while that in the inequality of Theorem 4 is taken over $1 \le j \le l$.

To prove Theorem 4, we will need several lemmas. The first one has its own interests. It is simple but decisive.

Lemma 3. *Let* $(\Omega, \mathcal{A}, \mu)$ *be a measure space. Let* $0 < \alpha < \beta < \infty$ *be two constants. There exists a constant* $0 < \gamma = \gamma(\alpha, \beta) < 1$ *such that the inequality*

$$\left| \int \phi \chi d\mu \right| \leq \gamma \int |\phi| \chi d\mu$$

holds for any measurable function χ *such that* $\alpha \leq \chi(x) \leq \beta$ *and any integrable function* ϕ *such that* $\int \phi d\mu = 0$ *(the optimal* γ *is* $(\beta - \alpha)/(\beta + \alpha)$*).*

Proof. The special case (corresponding to the discrete measure $\mu = \delta_1 + \delta_2$)

$$|x_1 - x_2| \leq \gamma(x_1 + x_2) \quad (\alpha \leq x_1, x_2 \leq \beta)$$

is trivial. Now without loss of generality, we assume that

$$\int_{\phi > 0} \phi d\mu = -\int_{\phi < 0} \phi d\mu = 1.$$

Apply the special case to

$$x_1 = \int_{\phi > 0} \phi \chi d\mu, \quad x_2 = -\int_{\phi < 0} \phi \chi d\mu.$$

Since $\alpha \leq x_1, x_2 \leq \beta$, we have

$$\left| \int \phi \chi d\mu \right| = |x_1 - x_2| \leq \gamma(x_1 + x_2) = \gamma \int |\phi| \chi d\mu. \qquad \square$$

The main idea of the proof of Theorem 4 is to introduce a sequence of linear operators $\mathcal{P} = \{P_n\}_{n=1}^{\infty}$ defined as

$$P_n \phi(x) = \tilde{\mathcal{L}}^n(f^n(x)) = \sum_{y \in f^{-n}(f^n(x))} \tilde{\psi}(y) \phi(y).$$

As we have seen in [FJ] that P_n is positive, $P_n 1 = 1$ and

$$P_m P_n = P_n P_m = P_m \quad (m \geq n \geq 1).$$

Assume for the moment $\lambda > 3$. Fix $n \geq 1$ and ϵ such that $0 < c_2 \epsilon \leq a$. Let $\mathcal{Q}_k$ $(1 \leq k \leq n)$ be the partitions constructed in Proposition 2. Let us still use $\mathcal{Q}_k$ to denote the σ-algebra generated by $\mathcal{Q}_k$. Let $\mathbb{E}_k = \mathbb{E}(\cdot | \mathcal{Q}_k)$ be the conditional expectation with respect to $\mathcal{Q}_k$ on the probability space (X, μ).

Lemma 4. *Let* $p_0 = p(\epsilon)$ *be a fixed integer in Proposition 1 (4). Then there exists a constant* $0 < \gamma < 1$ *depending only upon* ϵ *and* (f, ψ) *such that for any* $\phi \in L^{\infty}(\mu)$ *with* $\langle \mu, \phi \rangle = 0$*, any* $p \geq p_0$*, and any* $k \geq 1$*,*

$$\| P_{k+p} \mathbb{E}_k \phi \|_{\infty} \leq \gamma \| \phi \|_{\infty}.$$

Proof. Note that

$$P_{k+p}\mathbb{E}_k\phi(x) = \sum_{y\in f^{-(k+p)}(f^{k+p}(x))} \tilde{G}_{k+p}(y)\mathbb{E}_k\phi(y)$$
$$= \sum_{A\in\mathcal{Q}_k} \frac{\int_A \phi d\mu}{\mu(A)} \sum_{y\in A\cap f^{-(k+p)}(f^{k+p}(x))} \tilde{G}_{k+p}(y).$$

By Propositions 1 (4) and the Gibbs property in Theorem 1 (3), there is a constant $C_0 = C_0 > 0$ such that

$$C_0^{-1} \le \frac{1}{\mu(A)} \sum_{y\in A\cap f^{-(k+p)}(f^{k+p}(x))} \tilde{G}_{k+p}(y) \le C_0.$$

So Lemma 3 implies that we have a constant $0 < \gamma = (C_0 - C_0^{-1})/(C_0 + C_0^{-1}) < 1$ such that

$$|P_{k+p}\mathbb{E}_k\phi(x)| \le \gamma \sum_{A\in\mathcal{Q}_k} \frac{\int_A |\phi| d\mu}{\mu(A)} \sum_{y\in A\cap f^{-(k+p)}(f^{k+p}(x))} \tilde{G}_{k+p}(y)$$
$$\le \gamma\|\phi\|_\infty \sum_{A\in\mathcal{Q}_k} \sum_{y\in A\cap f^{-(k+p)}(f^{k+p}(x))} \tilde{G}_{k+p}(y) = \gamma\|\phi\|_\infty,$$

because $\sum_{A\in\mathcal{Q}_k} \sum_{y\in A\cap f^{-(k+p)}(f^{k+p}(x))} \tilde{G}_{k+p}(y) = P_{k+p}1 = 1.$ □

For any function ϕ defined on X, define

$$\mathcal{V}_m^{(n,\epsilon)}(\phi) = \sup_{A\in\mathcal{Q}_m} \sup_{x,y\in A} |\phi(x) - \phi(y)| \quad (1 \le m \le n).$$

This describes the variation of ϕ on the partition $\mathcal{Q}_m$ which depends on n and ϵ. A function ϕ is *$\mathcal{Q}_m$-measurable* if $\mathcal{V}_m^{(n,\epsilon)}(\phi) = 0$, i.e., ϕ is a piecewise constant function with respect to $\mathcal{Q}_m$.

Lemma 5. *For any ϵ such that $0 < c_2\epsilon \le a$, there exists a constant integer $q_0 \ge 1$ such that for any $q \ge q_0$ with $n \ge m \ge k + q > k \ge 1$ and any $\mathcal{Q}_k$-measurable ϕ we have*

$$\mathcal{V}_m^{(n,\epsilon)}(P_{k+q}\phi) \le K\|\phi\|_\infty\tilde{\omega}(\lambda^{-(m-k)}c_2\epsilon\lambda^q).$$

Proof. Suppose $A \in \mathcal{Q}_m$ and $x, y \in A$. Let $f^{-k}(f^k(x)) = \{x_j\}$ and $f^{-k}(f^k(y)) = \{y_j\}$. By Proposition 2, A is contained in a d_m-ball of radius $c_2\epsilon$. We may assume that for each j, x_j and y_j are contained in a d_m-ball of radius $c_2\epsilon$ which is contained in a d_{m-q}-ball of radius $c_2\epsilon\lambda^{-q}$. Take q_0 such that

$$\lambda^{q_0} > c_2/c_1.$$

Then x_j and y_j are contained in a d_{m-q}-ball of radius $c_1\epsilon$ which is contained in a d_k-ball of radius $c_1\epsilon$ because $m-q \geq k$. As ϕ is $\mathcal{Q}_k$-measurable, $\phi(x_j) = \phi(y_j)$. So,

$$\begin{aligned}
|P_{k+q}\phi(x) - P_{k+q}\phi(y)| &= \left| \sum_j \tilde{G}_{k+q}(x_j)\phi(x_j) - \sum_j \tilde{G}_{k+q}(y_j)\phi(y_j) \right| \\
&\leq \|\phi\|_\infty \sum_j \left| \tilde{G}_{k+q}(x_j) - \tilde{G}_{k+q}(y_j) \right| \\
&\leq \|\phi\|_\infty \sum_j \tilde{G}_{k+q}(y_j) \left| \frac{\tilde{G}_{k+q}(x_j)}{\tilde{G}_{k+q}(y_j)} - 1 \right| \\
&\leq K\|\phi\|_\infty \tilde{\omega}(\lambda^{-(m-k+q)} c_2 \epsilon) \sum_j \tilde{G}_{k+q}(y_j) \\
&\leq K\|\phi\|_\infty \tilde{\omega}(\lambda^{-(m-k)} c_2 \epsilon \lambda^q).
\end{aligned}$$

We have used the remark after Lemma 2 and the fact $\sum_j \tilde{G}_k(x_j) = P_k 1 = 1$. □

The following lemma is obvious for any μ-measurable function ϕ.

Lemma 6. *For* $1 \leq k \leq n$*, we have*

$$\|(I - \mathbb{E}_k)\phi\|_\infty \leq \mathcal{V}_k^{(n,\epsilon)}(\phi).$$

Proof of Theorem 4. Let p be the biggest of $p_0 = p(\epsilon)$ in Lemma 4 and of q_0 in Lemma 5. For $n \geq k + p$, we have $P_n = P_n P_{k+p}$. Write $Q_{k+p} = P_{k+p}\mathbb{E}_k$. We have

$$P_n = P_n(I - \mathbb{E}_k) + P_n\mathbb{E}_k = P_n(I - \mathbb{E}_k) + P_n Q_{k+p}.$$

By induction, we have

$$\begin{aligned}
P_n &= P_n(I - \mathbb{E}_{n_0}) + P_n Q_{n_0+p} \\
&= P_n(I - \mathbb{E}_{n_0}) + P_n[(I - \mathbb{E}_{n_1}) + Q_{n_1+p}]Q_{n_0+p} \\
&= P_n(I - \mathbb{E}_{n_0}) + P_n(I - \mathbb{E}_{n_1}) + P_n Q_{n_1+p} Q_{n_0+p} \\
&\quad \cdots \\
&= P_n\left[(I - \mathbb{E}_{n_0}) + \sum_{j=1}^{\ell} (I - \mathbb{E}_{n_j}) \prod_{i=0}^{j-1} Q_{n_i+p} + \prod_{i=0}^{\ell-1} Q_{n_i+p}\right].
\end{aligned}$$

By using the fact $\|P_n\phi\|_\infty \leq \|\phi\|_\infty$, Lemma 6, and Lemma 4, we have

$$\begin{aligned}
\|P_n\phi\|_\infty &\leq \|(I - \mathbb{E}_{n_0})\phi\|_\infty \\
&\quad + \sum_{j=1}^{\ell} \left\| (I - \mathbb{E}_{n_j}) \prod_{i=0}^{j-1} Q_{n_i+p}\phi \right\|_\infty + \left\| \prod_{i=0}^{\ell-1} Q_{n_i+p}\phi \right\|_\infty \\
&\leq \mathcal{V}_{n_0}^{(n,\epsilon)}(\phi) + \sum_{j=1}^{\ell} \mathcal{V}_{n_j}^{(n,\epsilon)}\left(\prod_{i=0}^{j-1} Q_{n_i+p}\phi \right) + \gamma^\ell \|\phi\|_\infty.
\end{aligned}$$

Let $\phi_j = \prod_{i=0}^{j-1} Q_{n_i+p}\phi$. Then

$$\phi_j = \prod_{i=0}^{j-1} Q_{n_i+p}\phi = Q_{n_{j-1}+p}\phi_{j-1} = P_{n_{j-1}+p}\mathbb{E}_{n_{j-1}}\phi_{j-1}.$$

Here $\tilde{\phi}_j = \mathbb{E}_{n_{j-1}}\phi_{j-1}$ is $Q_{n_{j-1}}$-measurable. From Lemma 5 and the fact $\|\tilde{\phi}_j\|_\infty \le \|\phi\|_\infty$, we get

$$\begin{aligned}\mathcal{V}_{n_j}^{(n,\epsilon)}\left(\prod_{i=0}^{j-1} Q_{n_i+p}\phi\right) &= \mathcal{V}_{n_j}^{(n,\epsilon)}(P_{n_{j-1}+p}\tilde{\phi}_j)\\ &\le K\|\phi\|_\infty \tilde{\omega}(\lambda^{-(n_j-n_{j-1})}c_2\epsilon\lambda^p).\end{aligned}$$

On the other hand,

$$\mathcal{V}_{n_0}^{(n,\epsilon)}(\phi) \le \sup_{d_{n_0}(x,y)\le c_2\epsilon} |\phi(x)-\phi(y)| \le \Omega(c_2\epsilon\lambda^{-n_0}).$$

Thus we obtain

$$\|P_n\phi\|_\infty \le \Omega_\phi(c_2\epsilon\lambda^{-n_0}) + K\|\phi\|_\infty \sum_{j=1}^{\ell} \tilde{\omega}(\lambda^{-(n_j-n_{j-1})}c_2\epsilon\lambda^p) + \gamma^\ell\|\phi\|_\infty.$$

Because of $P_n\phi(x) = (\tilde{\mathcal{L}}^n\phi)(f^n(x))$ and the surjectivity of f^n, we now get

$$\|\tilde{\mathcal{L}}^n\phi\|_\infty \le \Omega_\phi(c_2\epsilon\lambda^{-n_0}) + K\|\phi\|_\infty \sum_{j=1}^{\ell} \tilde{\omega}(\lambda^{-(n_j-n_{j-1})}c_2\epsilon\lambda^p) + \gamma^\ell\|\phi\|_\infty.$$

For general $\lambda > 1$, first take an integer $\theta > 1$ such that $\lambda^\theta > 3$ and consider the local expanding and mixing map $g = f^\theta$. Then consider the normalization $\tilde{\mathcal{L}}_g$ of the Ruelle operator $\mathcal{L}_g$. Then, there are constants $0 < \gamma_g < 1$, $p_g \ge 0$, $K_g > 0$ such that for any ϕ having mean value zero with respect to μ and any integer partition of $[1, n]$, $1 \le n_0 < n_1 < \cdots < n_{\ell-1} < n_\ell \le n$, satisfying $n_j - n_{j-1} > p_g$,

$$\|\tilde{\mathcal{L}}_g^n\phi\|_\infty \le \Omega_\phi(c_2\epsilon\lambda^{-\theta n}) + K_g\|\phi\|_\infty \sum_{j=1}^{\ell} \tilde{\omega}(\lambda_g^{-\theta(n_j-n_{j-1})}c_2\epsilon\lambda^p) + \gamma_g^\ell\|\phi\|_\infty.$$

This, together with the fact $\tilde{\mathcal{L}}_g^n = \tilde{\mathcal{L}}_f^{\theta n}$, implies the desired result. □

Proof of Theorem 3. The relations between $\mathcal{L}$ and $\tilde{\mathcal{L}}$ and between $\mathcal{L}^*$ and $\tilde{\mathcal{L}}^*$ are

$$\mathcal{L}^n\phi = \rho^n h\tilde{\mathcal{L}}^n(\phi h^{-1}) \quad \text{and} \quad \mathcal{L}^{*n}\nu = \rho^n h^{-1}\tilde{\mathcal{L}}^{*n}(h\nu).$$

Let $\mu = h\nu$. Then $\langle \nu, \phi\rangle = 0$ iff $\langle \mu, \phi h^{-1}\rangle = 0$. Therefore,

$$\|\rho^{-n}\mathcal{L}^n\phi\|_\infty \le \|h\|_\infty\|\tilde{\mathcal{L}}^n(\phi h^{-1})\|_\infty.$$

However, denoting $h_{\min} = \min_x h(x)$ we have

$$\Omega_{\phi h^{-1}}(t) \le (h_{\min})^{-2}\|\phi\|_\infty \Omega_h(t) + (h_{\min})^{-1}\Omega_\phi(t).$$

Notice that $h \in \mathcal{H}^{\tilde{\omega}}$. By Theorem 4, there is a constant $C > 0$ such that

$$\|\rho^{-n}\mathcal{L}^n\phi\|_\infty \le C\left(\Omega_\phi(c_2\epsilon\lambda^{-n_0}) + \|\phi\|_\infty \sum_{j=0}^{\ell} \tilde{\omega}(\lambda^{-(n_j-n_{j-1})}c_2\epsilon\lambda^p) + \gamma^\ell\|\phi\|_\infty\right).$$

□

4. Examples

If $\omega(t) \le Ct^\alpha$ for some constants $C > 0$ and $0 < \alpha \le 1$, then $\tilde{\omega}(t) \le \tilde{C}t^\alpha$ for another constant $\tilde{C} > 0$. In this case, $\mathcal{H}^\omega = \mathcal{H}^{\tilde{\omega}} = \mathcal{C}^\alpha$ is the α-Hölder continuous space and it is known that the convergence speed is exponential (cf. [Bo, PP]), i.e. there are constants $C > 0$ and $\vartheta > 0$ such that for any $\phi \in \mathcal{C}^\alpha$,

$$\|\rho^{-n}\mathcal{L}^n\phi - \langle\nu, \phi\rangle h\|_\infty \le Ce^{-\vartheta n} \quad (n \ge 1).$$

Moreover, $\mathcal{L} : \mathcal{C}^\alpha \to \mathcal{C}^\alpha$ is quasi-compact (see [PP, He]).

When ψ is less regular, $\|\rho^{-n}\mathcal{L}^n\phi - \langle\nu, \phi\rangle h\|_\infty$ for $\phi \in \mathcal{H}^\omega$ may not have exponential decay. Our result will show different speeds for the decay of

$$\|\rho^{-n}\mathcal{L}^n\phi - <\nu, \phi> h\|_\infty$$

when ω satisfies the Dini condition. Following are some illustrating examples.

Corollary 1. *Suppose* $\alpha, \beta > 1$ *and*

$$\omega(t) = \frac{1}{|\log t|^\beta} \quad \textit{and} \quad \omega_0(t) = \frac{1}{|\log t|^\alpha}.$$

Suppose $0 < \psi \in \mathcal{H}^\omega$ *is the potential and* $\phi \in \mathcal{H}^{\omega_0}$ *is any function such that* $\int \phi d\nu = 0$, *then there exists a constant* $C > 0$ *such that*

$$\|\rho^{-n}\mathcal{L}^n\phi\|_\infty \le C\frac{(\log n)^{\max\{\alpha,\beta\}}}{n^{\min\{\alpha,\beta-1\}}} \quad (n \ge 1).$$

Proof. Note that $\tilde{\omega}(t) = O(|\log t|^{\beta-1})$. Apply Theorem 2 by choosing

$$n_0 = n_j - n_{j-1} = \left[\frac{n}{\log n}\right] \quad (1 \le j \le \ell) \quad \text{with} \quad \ell = [c\log n] - 1,$$

where $[x]$ denotes the integral part of a real number x and $c > 0$ is chosen sufficiently large. We get

$$\begin{aligned}\|\rho^{-n}\mathcal{L}^n\phi\|_\infty &\le C'\left(\left(\frac{\log n}{n}\right)^\alpha + (\log n)\cdot\left(\frac{\log n}{n}\right)^{\beta-1} + \gamma^{c\log n}\right)\\ &\le C\frac{(\log n)^{\max(\alpha,\beta)}}{n^{\min(\alpha,\beta-1)}},\end{aligned}$$

where $C', C > 0$ are two constants. □

Corollary 2. *Suppose* $\omega(t) = e^{-\alpha|\log\log t|^\beta}$ $(\alpha > 0, \beta > 1)$. *Suppose* $\psi \in \mathcal{H}^\omega$ *is the potential and* $\phi \in \mathcal{H}^\omega$ *is any function with* $\int \phi d\nu = 0$. *Then for any* $\varepsilon > 0$ *there exists* $B = B(\alpha, \beta, \varepsilon) > 0$ *such that*

$$\|\rho^{-n}\mathcal{L}^n\phi\|_\infty \leq Be^{-(\alpha-\varepsilon)(\log n)^\beta}, \quad (n \geq 1).$$

Proof. We show first the estimate (t being small)

$$\tilde{\omega}(t) \leq \int_0^t e^{-\alpha(\log\log\frac{1}{\xi})^\beta}\frac{d\xi}{\xi} \leq C\frac{|\log t|}{|\log\log\frac{1}{t}|^{\beta-1}} \cdot e^{-\alpha|\log\log\frac{1}{t}|^\beta}.$$

In fact, by making successively two changes of variables $u = |\log\xi|$ and $v = u/|\log t|$, we get

$$\begin{aligned}\tilde{\omega}(t) &\leq \int_{|\log t|}^\infty e^{-\alpha|\log u|^\beta}du \\ &= |\log t|\int_1^\infty e^{-\alpha(\log v + (\log\log\frac{1}{t})^\beta}dv.\end{aligned}$$

By using the inequality $(1+x)^\beta \geq 1 + \beta x$ $(\beta \geq 1, x \geq 0)$, we get

$$\tilde{\omega}(t) \leq |\log t| \cdot e^{-\alpha(\log\log\frac{1}{t})^\beta} \cdot \int_1^\infty e^{-\beta\alpha(\log\log\frac{1}{t})^{\beta-1}\log v}dv.$$

Now note that the integrand in the last integral is actually a polynomial and its integral is equal to

$$\left(\beta\alpha(\log\log\frac{1}{t})^{\beta-1} - 1\right)^{-1}.$$

Apply Theorem 2 by choosing

$$n_0 = n_j - n_{j-1} = \left[\frac{n}{\log^q n}\right] \quad (1 \leq j \leq \ell) \quad \text{with} \quad \ell = \left[\log^q n\right] - 1,$$

where $q > \beta$. We get that, up to a multiplicative constant, $\|\rho^{-n}\mathcal{L}^n\phi\|_\infty$ is bounded by the sum of the following three terms:

$$e^{-\alpha(\log n - q\log\log n)^\beta},$$

$$\log^q n \cdot \frac{n - \log^q n}{(\log n - q\log\log n)^{\beta-1}} \cdot e^{-\alpha(\log n - q\log\log n)^\beta},$$

and

$$e^{\log\gamma\log^q n}.$$

To finish the proof, it suffices to note that the second is the biggest and it is bounded up to a constant by $Be^{-(\alpha-\epsilon)(\log n)^\beta}$. $\square$

Corollary 3. *Suppose* $\omega(t) = e^{-\alpha|\log t|^\beta}$ $(\alpha > 0, 0 < \beta < 1)$. *Suppose* $\psi \in \mathcal{H}^\omega$ *is the potential and* $\phi \in \mathcal{H}^\omega$ *is any function with* $\int \phi d\nu = 0$. *Then there exists* $B = B(\alpha, \beta) > 0$ *and* $C = C(\alpha, \beta) > 0$ *such that*

$$\|\rho^{-n}\mathcal{L}^n\phi\|_\infty \leq Be^{-Cn^{\frac{\beta}{1+\beta}}}, \quad (n \geq 1).$$

Proof. We show first the estimate

$$\tilde{\omega}(t) \le \int_0^t e^{-\alpha|\log\xi|^\beta}\frac{d\xi}{\xi} \le |\log t|^{1-\beta}\cdot e^{-\alpha|\log t|^\beta}.$$

By making the change of variables $u = |\log\xi|$, we get

$$\tilde{\omega}(t) \le \int_{|\log t|}^\infty e^{-\alpha u^\beta}du.$$

So, it suffices to show that for any $R > 0$,

$$\int_R^\infty e^{-\alpha u^\beta}du \le CR^{1-\beta}e^{-\alpha R^\beta}.$$

Observe that for $z \ge 1$,

$$\int_z^\infty e^{-x^2}xdx = \frac{1}{2}e^{-z^2}$$
$$\int_z^\infty e^{-x^2}x^a dx = \frac{1}{2}z^{a-1}e^{-z^2} + \frac{a-1}{2}\int_z^\infty e^{-x^2}x^{a-2}dx.$$

Let $a \ge 1$ and let q be the smallest integer such that $a - 2q \le 1$. Applying q times the last equality enables us to get

$$\begin{aligned}\int_z^\infty e^{-x^2}x^a dx &\le C\left(z^{a-1}e^{-z^2} + \int_z^\infty e^{-x^2}x^{a-2q}dx\right)\\ &\le C\left(z^{a-1}e^{-z^2} + \int_z^\infty e^{-x^2}xdx\right)\\ &\le C'z^{a-1}e^{-z^2}.\end{aligned}$$

Now to obtain the claimed inequality, it suffices to apply the above inequality to the right-hand side of the following equality:

$$\int_R^\infty e^{-\alpha u^\beta}du = \frac{2}{\beta\alpha^{\frac{1}{\beta}}}\int_{\sqrt{\alpha R^\beta}}^\infty e^{-x^2}x^{\frac{2}{\beta}-1}dx.$$

Apply Theorem 2 by choosing

$$n_0 = n_j - n_{j-1} = \left[n^{\frac{1}{1+\beta}}\right] \quad (1 \le j \le \ell) \quad \text{with} \quad \ell = \left[n^{\frac{\beta}{1+\beta}}\right] - 1.$$

Then up to a multiplicative constant, $\|\rho^{-n}\mathcal{L}^n\phi\|_\infty$ is bounded by

$$e^{-\alpha n^{\frac{\beta}{1+\beta}}} + n^{1-\beta}n^{\frac{1-\beta}{1+\beta}}e^{-\alpha n^{\frac{\beta}{1-\beta}}} + e^{\log\gamma\cdot n^{\frac{\beta}{1+\beta}}}.$$

It is clear that each of the above three terms is bounded by Ce^{-Bn^β} when $B > \max(\alpha, |\log\gamma|)$. □

5. Applications

5.1. Correlations. Suppose (X, d) is a compact metric space and f is an expanding and mixing dynamical system on X. Suppose ψ is a potential function in $\mathcal{H}^\omega$, where ω is a modulus of continuity satisfying the Dini condition. Let $\mu = h\nu_\psi$ be the Gibbs measure associate to ψ. Then $\tilde{\mathcal{L}}^*\mu = \mu$ and μ is f-invariant (for $\tilde{\mathcal{L}}(\phi \circ f) = \phi$). For a continuous function $\phi \in \mathcal{C}$, $\phi \circ f^n$ is a stationary process defined on the probability space (X, μ). Its *correlation* is defined by

$$\Phi(n) = \int (\phi \circ f^n)\phi d\mu - \left(\int \phi \, d\mu\right)^2.$$

We have

Theorem 5. *Under the same condition as in Theorem 2,*

$$|\Phi(n)| \le C\|\phi\|_\infty \left(\Omega_\phi(c_2\epsilon\lambda^{-n_0}) + \|\phi\|_\infty \sum_{j=0}^{\ell} \omega(\lambda^{-(n_j - n_{j-1})} c_2 \epsilon \lambda^p) + \|\phi\|_\infty \gamma^\ell\right),$$

where $1 \le n_0 < n_1 < \cdots < n_\ell \le n$ *with* $n_j - n_{j-1} > p$ *and* $C > 0$ *is a constant.*

Proof. Let $\tilde{\phi} = \phi - \langle \mu, \phi\rangle$. Then $\int \tilde{\phi} d\mu = 0$ and

$$\Phi(n) = \int (\tilde{\phi} \circ f^n)\tilde{\phi} d\mu = \langle \mu, (\tilde{\phi} \circ f^n)\tilde{\phi}\rangle.$$

But

$$\langle \mu, (\tilde{\phi} \circ f^n)\tilde{\phi}\rangle = \langle \tilde{\mathcal{L}}^{*n}\mu, (\tilde{\phi} \circ f^n)\tilde{\phi}\rangle = \langle \mu, \mathcal{L}^n((\tilde{\phi} \circ f^n)\tilde{\phi})\rangle = \langle \mu, \tilde{\phi}\mathcal{L}^n\tilde{\phi}\rangle.$$

So $|\Phi(n)| \le \|\tilde{\phi}\|_\infty \|\tilde{\mathcal{L}}^n\tilde{\phi}\|_\infty$. Thus the claimed result follows from Theorem 2. □

5.2. Central limit theorem. The other way to describe the statistical properties of a dynamical system is the central limit theorem. For expanding and mixing dynamical systems, the central limit theorem holds thanks to Theorem 2.

Theorem 6. *Let*

$$\omega(t) = \frac{1}{|\log t|^{2+\epsilon}} \quad \text{and} \quad \omega_0(t) = \frac{1}{|\log t|^{1+\epsilon}} \quad (\epsilon > 0).$$

Suppose $0 < \psi \in \mathcal{H}^\omega$ *is a potential and* $\mu = h\nu_\psi$ *is the Gibbs measure associate to* ψ *(*μ *is* f*-invariant). For any* $\phi \in \mathcal{H}^{\omega_0}$*, we have*

$$\lim_{n\to\infty} \mu\left\{x : \sum_{j=0}^{n-1} \phi \circ f^j - n\int \phi d\mu \le t\sqrt{n}\right\} = \frac{1}{\sqrt{2\pi}\sigma}\int_{-\infty}^{t} \exp\left(-\frac{x^2}{2\sigma^2}\right) dx,$$

where $\sigma^2 = -\mathbb{E}\phi^2 + 2\sum_{j=0}^{\infty} \mathbb{E}(\phi \cdot \phi \circ f^j)$, $\mathbb{E}\phi$ *denoting* $\langle \mu, \phi\rangle$.

Proof. Without loss of generality, assume $\int \phi d\mu = 0$. Let $\mathcal{B}$ be the Borel σ-field. For $n \geq 1$, let $\mathcal{B}_n = f^{-n}\mathcal{B}$. Define $V\phi = \phi \circ f$ for $\phi \in L^2(\nu)$. Let V^* be the adjoint operator of $V : L^2 \to L^2$. By Theorem 1.1 of [Li], it suffices to show the convergences of the following two series:

$$\sum_{n=0}^{\infty} |\mathbb{E}(\phi V^n \phi)| < \infty, \quad \sum_{n=0}^{\infty} \mathbb{E}|V^{*n}\phi| < \infty.$$

Since $\tilde{\mathcal{L}}^* \mu = \mu$,

$$\mathbb{E}(\phi V^n \phi) = \langle \mu, \phi \cdot V^n \phi \rangle = \langle \tilde{\mathcal{L}}^{*n} \mu, \phi \cdot V^n \phi \rangle = \langle \mu, \mathcal{L}^n(\phi \cdot V^n \phi) \rangle = \langle \mu, \phi \mathcal{L}^n \phi \rangle.$$

So

$$|\mathbb{E}(\phi V^n \phi)| \leq \|\phi\|_\infty \|\tilde{\mathcal{L}}^n \phi\|_\infty.$$

Then, by Corollary 1, we have

$$|\mathbb{E}(\phi V^n \phi)| = O\left(\frac{(\log n)^{2+\epsilon}}{n^{1+\epsilon}}\right).$$

Thus we get the convergence of the first series. On the other hand, observe that $V^*\phi = \tilde{\mathcal{L}}\phi$. So

$$\mathbb{E}|V^{*n}\phi| \leq \|\tilde{\mathcal{L}}^n \phi\|_\infty = O\left(\frac{(\log n)^{2+\epsilon}}{n^{1+\epsilon}}\right).$$

The convergence of the second series follows. □

Acknowledgement. This work was started when the second author visited the Faculté de Mathématiques et Informatique at the Université de Picardie Jules Verne in Amiens, France and it was done when the first author visited the Institute of Mathematical Science (IMS) at the Chinese University of Hong Kong and when the second author visited the Institute for Mathematical Research (FIM) at ETH-Zürich in Switzerland. The authors would like to thank these institutes for hospitality and support. The authors also would like to thank Professors A.-S. Sznitman, K. S. Lau and O. Lanford III for their interest and helpful discussions. Thanks also go to A. Rivière et Y. L. Ye for valuable comments.

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Communicated by Ya. G. Sinai