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ε-Pisot numbers in any real algebraic number field are relatively dense

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Abstract

An algebraic integer is called an ε-Pisot number ($\varepsilon > 0$) if its Galois conjugates have absolute value less then ε. Let K be any real algebraic number field. We prove that the subset of K consisting of ε-Pisot numbers which have the same degree as that of the field is relatively dense in the real line $\mathbb{R}$. This has some applications to non-stationary products of random matrices involving Salem numbers.

Keywords: Real algebraic number fields; PV-numbers; Salem numbers

1. Introduction

We are dealing with special algebraic integers, including Pisot numbers and Salem numbers. For basic definitions and results of algebraic numbers we suggest the book [4] and for a survey on Pisot numbers and Salem numbers we suggest the book [6].

Definition 1.1. Let $\varepsilon > 0$ be given. An algebraic integer is called an *ε-Pisot number* if all its Galois conjugates have modulus less than ε. The usual *Pisot numbers* correspond to 1-Pisot numbers.

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Definition 1.2. An algebraic integer is called a *Salem number* if all its Galois conjugates have modulus at most 1 and at least one conjugate lies on the unit circle.

Definition 1.3. A subset E of real numbers is said to be *relatively dense* if there is a number L such that for any $x \in \mathbb{R}$ we have $E \cap [x, x+L] \neq \emptyset$.

Our main result is the following.

Theorem 1.4. *Let K be a real algebraic number field. Then for any $\varepsilon > 0$ the set of ε-Pisot numbers in K which have the same degree as that of the field K is relatively dense in $\mathbb{R}$.*

Recall that a real algebraic number field K of degree d is an extension of the field $\mathbb{Q}$ of rational numbers by d independent real algebraic numbers. It is also an extension of $\mathbb{Q}$ by a single algebraic number of degree d.

A Salem number τ is always of even degree say $2d$. It has exactly one real Galois conjugate inside the unit circle and all other Galois conjugates are pairwise complex conjugates on the unit circle. We can write them as $\tau, 1/\tau, e^{i\alpha_1}, e^{-i\alpha_1}, \ldots, e^{i\alpha_{d-1}}, e^{-i\alpha_{d-1}}$ where $\alpha_i \in \mathbb{R}$ for $i = 1, \ldots, d-1$.

Corollary 1.5. *Let τ be a Salem number then for any $\varepsilon > 0$ the set of $\lambda \in \mathbb{R}$ such that $\|\lambda\tau^n\| < \varepsilon$ for all $n \in \mathbb{N}$ is relatively dense in $\mathbb{R}$, where $\|x\|$ denotes the distance to the rational integers of the real number x.*

Let $\tau > 1$ be a Salem number and let $M : \mathbb{R} \to \mathrm{GL}(\mathbb{R}^{\mathrm{d}})$ be a 1-periodic matrix-valued Lipschitz function. Consider the matrix products

$$P_n(x) = M\left(\tau^{n-1}x\right) \cdots M(\tau x)M(x).$$

Corollary 1.6. *Suppose that the entries of M are either identically zero or strictly positive. Then the following limit exists for almost all point $x \in \mathbb{R}$ with respect to the Lebesgue measure and is independent of x*

$$\lim_{n\to\infty} \frac{1}{n} \log\left\| P_n(x)\right\|$$

where $\|A\|$ denotes the norm of a matrix A.

Corollary 1.6 is a consequence of Corollary 1.5 and the following Kingman type theorem proved in [1].

Theorem 1.7. *Let $(u_n)_{n\geqslant 0}$ be a totally Bohr ergodic sequence of real numbers and $(f_n)_{n\geqslant 0}$ be a sequence of uniformly almost periodic functions. Suppose*

(i) *The sequence $(n^{-1} f_n)$ has joint periods.*

(ii) *The following subadditivity is fulfilled*

$$f_{n+m}(x) \leqslant f_n(x) + f_m(u_n x) \quad \text{for a.e. } x \text{ and all } n, m.$$

(iii) *For any* $m \geqslant 1$ *there exists* $g_m(x) \in L^\infty(\mathbb{R})$ *such that*

$$f_n(u_m x) \leqslant f_n(x) + g_m(x) \quad \text{for a.e. } x \text{ and all } n.$$

Then the following limit exists

$$\lim_{n\to\infty} \frac{1}{n} f_n(x) = \inf_n \frac{1}{n} \mathbb{M} f_n \quad \text{for a.e. } x.$$

We will explain in the proof the meaning of joint periods. Here we just recall that the sequence (β^k) with $\beta > 1$ is totally Bohr ergodic and that $\mathbb{M} f$ denotes the Bohr mean of a uniformly almost periodic function f.

If τ is an integer, Corollary 1.6 is well known as a special case of a theorem of Furstenberg–Kesten [2]. If τ is a Pisot number, Corollary 1.6 is obtained in [1].

To some extent, the result for Salem numbers in Corollary 1.5 is optimal in the sense that ε cannot be replaced by a sequence ε_n tending to zero, even for a single λ. In fact, otherwise $\lambda\tau^n$ has zero as limit (on the torus). However $\lambda\tau^n$ is dense in some interval, so the limit doesn't exist (see [6, p. 33]). Of course, for a Pisot number θ, $\|\lambda\theta^n\|$ is exponentially small as a function of n, for a fixed λ in the field of θ. The novelty of Corollary 1.5, even for Pisot numbers, is the uniform estimate (uniform for a relatively dense set of λ). Actually, the proof of Corollary 1.5 will show that for any $\varepsilon > 0$, the set of λ such that $\|\lambda\theta^n\| \leqslant \varepsilon r^n$ $(\forall n \geqslant 1)$ is relatively dense, where $0 < r < 1$.

2. Proofs

Proof of Theorem 1.4. Assume that the degree of K is d. Thus the field K admits a basis of algebraic integers

$$\omega_1, \ldots, \omega_d.$$

We denote their Galois conjugates by (the ith column corresponding to the conjugates of ω_i)

$$\begin{matrix} \omega_1^{(1)}, & \ldots, & \omega_d^{(1)} \\ \omega_1^{(2)}, & \ldots, & \omega_d^{(2)} \\ \vdots & \ddots & \vdots \\ \omega_1^{(d-1)}, & \ldots, & \omega_d^{(d-1)}. \end{matrix}$$

Consider the following system of inequalities whose unknown are rational integers $(m_1, m_2, \ldots, m_d) \in \mathbb{Z}^d$:

$$\begin{aligned}
&\left|m_1\omega_1^{(1)} + \cdots + m_d\omega_d^{(1)}\right| < \varepsilon,\\
&\left|m_1\omega_1^{(2)} + \cdots + m_d\omega_d^{(2)}\right| < \varepsilon,\\
&\qquad\vdots\\
&\left|m_1\omega_1^{(d-1)} + \cdots + m_d\omega_d^{(d-1)}\right| < \varepsilon.
\end{aligned}$$

To each solution $(m_1, m_2, \ldots, m_d)$, if any, of the system we associate the algebraic integer

$$\lambda(m_1, m_2, \ldots, m_d) = m_1\omega_1 + \cdots + m_d\omega_d.$$

The number $\lambda(m_1, m_2, \ldots, m_d)$ admits its Galois conjugates $m_1\omega_1^{(i)} + \cdots + m_d\omega_d^{(i)}$, $i = 1, \ldots, d-1$. So, it is an ε-Pisot number. We will conclude the proof by showing that the set of all numbers $\lambda(m_1, m_2, \ldots, m_d)$ is relatively dense in $\mathbb{R}$.

Indeed, the set of points $(x_1, \ldots, x_d) \in \mathbb{R}^d$ fulfilling the inequalities

$$\begin{aligned}
&\left|x_1\omega_1^{(1)} + \cdots + x_d\omega_d^{(1)}\right| < \varepsilon,\\
&\left|x_1\omega_1^{(2)} + \cdots + x_d\omega_d^{(2)}\right| < \varepsilon,\\
&\qquad\vdots\\
&\left|x_1\omega_1^{(d-1)} + \cdots + x_d\omega_d^{(d-1)}\right| < \varepsilon
\end{aligned}$$

is a thin "tube" around the line l parameterized as $\{(\alpha_1 t, \ldots, \alpha_d t)\colon t \in \mathbb{R}\}$ whose direction $(\alpha_1, \alpha_2, \ldots, \alpha_d)$ is given by

$$\alpha_1\omega_1^{(i)} + \cdots + \alpha_d\omega_d^{(i)} = 0 \qquad (1 \leqslant i \leqslant d-1). \tag{1}$$

Now the question is how to show that there are $(m_1, m_2, \ldots, m_d) \in \mathbb{Z}^d$ arbitrarily close to the line l such that $\lambda(m_1, m_2, \ldots, m_d)$ are relatively dense in the tube.

To this end, we consider the translation $T : \mathbb{T}^d \to \mathbb{T}^d$ defined by

$$T(t_1, t_2, \ldots, t_d) = (t_1 + \alpha_1, t_2 + \alpha_2, \ldots, t_d + \alpha_d).$$

The question will be reformulated in the following way: how to find a relatively dense set of n such that

$$T^n 0 \in U_{\varepsilon'}(0)$$

for any pre-described ε', where $U_{\varepsilon'}(0)$ is the ε'-ball of $\mathbb{T}^d$ centered at 0.

To prove this, we will use a basic fact about translations of the d-dimensional torus (actually a fact about group translations with respect to the Haar measure).

Lemma 2.1 (see [3, Proposition 4.3.3]). *Let T be a translation of $\mathbb{T}^d$ and $x \in \mathbb{T}^d$. Then the restricted action of T on the orbit closure $\overline{\{T^n x\colon n \in \mathbb{Z}\}}$ is uniquely ergodic.*

Since the unique ergodicity implies the uniform convergence of the Birkhoff means of continuous functions (see [3, Theorem 4.3.1]), we have for any elementary basic set B (in particular ε'-neighborhoods) that the "hitting sequence" $\{n \in \mathbb{Z}\colon\ T^n x \in B\}$ is relatively dense in $\mathbb{Z}$ and hence in $\mathbb{R}$ for an uniquely ergodic action T. We may also refer to [5].

Apply this to our case by choosing $B = U_{\varepsilon'}(0)$ with

$$\varepsilon' \max_{1 \leqslant i < d} \sum_{j=1}^{d} \bigl|\omega_j^{(i)}\bigr| < \varepsilon.$$

Let $\mathcal{Z}(\varepsilon')$ be the set of $n \in \mathbb{Z}$ having the property that there exists $(m_1, m_2, \ldots, m_d) \in \mathbb{Z}^d$ such that

$$|\eta_j| < \varepsilon', \quad \text{with } \eta_j := n\alpha_j - m_j \ (j = 1, 2, \ldots, d). \tag{2}$$

On one hand, by (1) and (2), for any $1 \leqslant i \leqslant d-1$ we have

$$\left|\sum_{j=1}^{d} m_j \omega_j^{(i)}\right| = \left| n \sum_{j=1}^{d} \alpha_j \omega_j^{(i)} - \sum_{j=1}^{d} \eta_j \omega_j^{(i)} \right| \leqslant \varepsilon' \sum_{j=1}^{d} \bigl|\omega_j^{(i)}\bigr| < \varepsilon,$$

and on the other hand we have

$$\lambda(m_1, m_2, \ldots, m_d) = n \sum_{j=1}^{d} \alpha_j \omega_j + \mathrm{O}(\varepsilon). \tag{3}$$

By Lemma 2.1, $\mathcal{Z}(\varepsilon')$ is relatively dense. Let $a = \sum_{j=1}^{d} \alpha_j \omega_j$. Since $a \neq 0$, the set $a\mathcal{Z}(\varepsilon')$ is relatively dense and so is the set of the all $\lambda(m_1, m_2, \ldots, m_d)$ because of (3). □

Proof of Corollary 1.5. Let τ be a Salem number of degree $2d$ and K its algebraic number field. For fixed $\varepsilon > 0$, the set of ε-Pisot numbers in K of degree $2d$ is relatively dense in $\mathbb{R}$. For such an ε-Pisot number β with its conjugates $\beta^{(j)}$ $(1 \leqslant j \leqslant 2d-1)$ we have

$$\beta\tau^n + \frac{\beta^{(1)}}{\tau^n} + \beta^{(2)} e^{i\alpha_1} + \beta^{(3)} e^{-i\alpha_1} + \cdots + \beta^{(2d-2)} e^{i\alpha_{d-1}} + \beta^{(2d-1)} e^{-i\alpha_{d-1}} \in \mathbb{Z}.$$

Therefore

$$\bigl\|\beta\tau^n\bigr\| < \varepsilon(2d-1) \quad \text{for all } n \in \mathbb{N}.$$

This proves the corollary. □

Proof of Corollary 1.6. For a non-negative matrix A and a fixed positive vector v, the matrix norm $\|A\|$ has the same size as the vector norm $|Av|$ (one is bounded by the other up to a multiplicative constant. We will use the norm $|v| = \sum_{j=1}^{d} |v_j|$ for $v = (v_1, \ldots, v_d)$). So, it suffices to show the limit of $n^{-1} f_n(x)$ is a constant where

$$f_n(x) = \log\bigl|M\bigl(\tau^{n-1}x\bigr)\cdots M(\tau x)M(x)v\bigr|.$$

By Theorem 1.7 (Theorem 2.5 in [1]), we have only to show that for any $\varepsilon > 0$ the set of $\lambda \in \mathbb{R}$ such that

$$n^{-1}\bigl|f_n(x+\lambda) - f_n(x)\bigr| \leqslant \varepsilon \quad (\forall x \in \mathbb{R},\ \forall n \geqslant 1) \tag{4}$$

is relatively dense (such a λ is called an ε-period of $n^{-1} f_n$ in [1]). Since $M(x)$ is Lipschitzian, so is $|M(x)v|$. Then, by Corollary 1.5, there is a relatively dense set of λ such that

$$\bigl|M\bigl(x + \lambda\tau^k\bigr)v - M(x)v\bigr| \leqslant C\varepsilon \tag{5}$$

for some constant C, all $x \in \mathbb{R}$ and all $k \geqslant 1$. On the other hand we have the following distortion lemma, which is easy to prove (see [1])

$$\log \frac{|M(x_1)M(x_2)\cdots M(x_n)v|}{|M(y_1)M(y_2)\cdots M(y_n)v|} \leqslant \left(\delta^{-1}\sum_{k=1}^{n}|x_k - y_k|\right) \tag{6}$$

where δ is the minimal value of those strictly positive entries of M. Both (5) and (6) together give

$$\bigl|f_n(x+\lambda) - f_n(x)\bigr| \leqslant C\delta^{-1}n\varepsilon.$$

This is (4) with ε replaced by $C\delta^{-1}\varepsilon$. Recall that $\varepsilon > 0$ is arbitrary. □

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