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Non-homogeneous equilibrium states and convergence speeds of averaging operators

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Abstract

We introduce non-homogeneous equilibrium states which include the classical equilibrium states for subshifts of finite type, Riesz products and G-measures. We prove a rather precise estimate for $\|P_n f\|_\infty$ where P_n are the averaging operators. Applications are given to estimate $\|L_g^n f\|_\infty$ (L_g being a transfer operator on a subshift of finite type) and then to prove a central limit theorem for $f(T^n x)$ (T being the shift), to study the almost everywhere convergence of some lacunary series with respect the equilibrium state and then to compute the Hausdorff dimension of the equilibrium state etc.

Introduction

The thermodynamical formalism was introduced by Ruelle [**20**] and Sinai [**21**]. It helps to understand the equilibrium state associated to a potential (or an interaction) in statistical physics. The existence and properties of equilibrium states were studied for different types of dynamical systems [**2**]. By using the Markov partitions, many systems can be reduced to a symbolic dynamical system called subshift of finite type. In this paper, we shall study the speed of convergence towards the equilibrium state associated to a potential of summable variation. More precisely, let L_g be the transfer operator defined by a potential of summable variation [**2**] (a definition will be given below). It is well known that there exists a unique equilibrium state μ associated to g and that for any continuous function f, $L_g^n f$ converges uniformly to a constant. It is the speed of this convergence that we would like to study.

The well-known classical result states that if g and f are Hölder continuous, the speed is exponential [**2**]. Our aim is to get sub-exponential speeds for less regular functions g and f. There are works done in this direction [**5**, **13**, **15**, **17**, **23**], which have interesting consequences both in dynamical systems and stochastic processes. The projective metric introduced by Birkhoff [**1**] are often used as a powerful tool [**12**, **15**]. A coupling approach is also used [**3**]. In this paper, we shall present a simple way to study this problem by using some ideas coming from [**6**, **9**, **10**]. Actually, our setting of the so-called non-homogeneous symbolic space is more general than that of subshift of finite type. The Riesz products in harmonic analysis [**24**] and G-measures [**4**] are thus included as special cases.

Our main result (Theorem 2) reads as an inequality. We state it here for the case of subshift of finite type (Theorem 3):

$$\|L_g^N f\|_\infty \leqslant A \left(\mathrm{var}_n(f) + \sum_{i=1}^{k} \sum_{j=n_{i+1}-n_i+1}^{n_{i+1}} \mathrm{var}_j(\log g) + \gamma^k \right)$$

for any $1 \leqslant n_0 < n_1 < \cdots < n_k \leqslant N$, where $\mathrm{var}_n(f)$ is the variation of a function f. We should point out that such a L^∞-estimate is the first one in the case of non-Hölder potentials and it is stronger than ever obtained estimates on the correlation which are essentially L^2-estimates.

Two applications will be given. One, consequence of Theorem 3, is to prove the central limit theorem (CLT) for the process $f(T^n x)$ with $\mathrm{var}_n(f) = O(n^{-1-\epsilon})$ ($\epsilon > 0$) relative to the Gibbs measure of a potential φ with $\mathrm{var}_n(\varphi) = O(n^{-2-\epsilon})$. The other, consequence of Theorem 2, is to prove the quasi-orthogonality of some systems of functions with respect to the non-homogeneous equilibrium state, which allows us to compute the dimension of the non-homogeneous equilibrium state.

In Section 1 we define the non-homogeneous equilibrium states and state the main results. In Section 2 we give a study of the existence and uniqueness of the non-homogeneous equilibrium states. The main result concerning the convergence speed is proved in Section 3. More precise results for subshifts of finite type are given in Section 4 and consequently we get a CLT under a weak regularity condition. In Section 5, the almost everywhere convergence of lacunary series is studied and we use the result to compute the Hausdorff dimension of the equilibrium state in Section 6. This dimension result improves those in [**7**, **10**, **18**].

1. *Definitions and main results*

Let $\{S_n\}_{n\geqslant 1}$ be a sequence of finite sets of discrete topology. Suppose $\ell_j = \mathrm{Card}\, S_j \geqslant 2$. Consider the infinite product space $X = \prod_{n=1}^{\infty} S_n$ equipped with the product topology. A compatible metric on X may be defined as $d(x, y) = (\ell_1 \ell_2 \cdots \ell_n)^{-1}$ with $n := n(x, y) = \sup\{j \geqslant 1 \colon x_i = y_i, \forall 1 \leqslant i \leqslant j\}$ (with convention $\sup \varnothing = 0$). Let $A = \{A_n\}_{n\geqslant 1}$ be a sequence of matrices such that $A_n \in M_{S_n \times S_{n+1}}$, that means the rows of A are indexed by S_n and the columns by S_{n+1}. Suppose the entries of A_n are 0 or 1. Such matrix is called an *incidence matrix*. We define a subspace X_A of X by

$$X_A = \{x = (x_j) \in X \colon A_n(x_n, x_{n+1}) = 1, \quad \forall n \geqslant 1\}.$$

We call X_A a *non-homogeneous symbolic space* (restricted by $A = \{A_n\}$). In the sequel, we always suppose that there exists an integer $M \geqslant 0$ such that

$$\prod_{j=n}^{n+M} A_j > 0 \quad (\forall n \geqslant 1) \tag{1}$$

($A > 0$ means that the entries of A are all positive). In this case, X_A is said to be *transitive*. Note that $X_A = X$ if all entries of every A_n are equal to 1. We call X the *full symbolic space*.

A sequence $G = \{g_n\}_{n\geqslant 1}$ of non-negative functions defined on X_A is called a *sequence of potentials* if for any $n \geqslant 1$, $g_n(x)$ does not depend on the first $n-1$ coordinates of x (so, we sometimes write $g_n(x) = g_n(x_n x_{n+1}\cdots)$). Further, it is said to be *normalized* if for any $n \geqslant 1$,

$$\sum_{y_n:\,A_n(y_n,x_{n+1})=1} g_n(y_n x_{n+1}\cdots) = 1 \quad (\forall x = (x_n) \in X_A). \tag{2}$$

For $n \geqslant 1$, let

$$G_n(x) = \prod_{j=1}^{n} g_n(x).$$

Then define a sequence of *averaging operators* $P_n\colon C(X_A) \to C(X_A)$, where $C(X_A)$ is the space of all continuous functions on X_A, by

$$P_n f(x) = \sum_{y_1,\cdots,y_n} G_n(y_1\cdots y_n x_{n+1}\cdots) f(y_1\cdots y_n x_{n+1}\cdots), \tag{3}$$

where the sum is taken over all sequences $(y_1,\ldots,y_n)$ such that

$$A_1(y_1,y_2)\cdots A_{n-1}(y_{n-1},y_n)A_n(y_n,x_{n+1}) = 1.$$

It will be checked that P_n is positive and $P_n 1 = 1$. Therefore, the adjoint operator $P_n^*\colon M_1^+(X_A) \to M_1^+(X_A)$ admits fixed points, where $M_1^+(X_A)$ is the space of all Borel probability measures on X_A. A measure $\mu \in M_1^+(X_A)$ is called a (non-homogeneous) *equilibrium state* associated to $G = \{g_n\}$ if $P_n^*\mu = \mu$ for all $n \geqslant 1$.

Consider two special cases. The first is the (classical) subshift of finite type. Suppose $S_n = S$ for all $n \geqslant 1$ where $S = \{1,2,\ldots,\ell\}$. Suppose $A_n = A$ for all $n \geqslant 1$ where A is a $\ell\times\ell$ incidence matrix. Then we get the well-known subshift of finite type X_A [**2**]. We will denote it by Σ_A in order to distinguish it from X_A in the general case. Note that Σ_A is transitive if and only if A is a primitive matrix. A specific property of Σ_A is that there is a shift $T\colon \Sigma_A \to \Sigma_A$ which is defined by $x = (x_n)_{n\geqslant 1} \to Tx = (x_{n+1})_{n\geqslant 1}$. We thus have a dynamical system (Σ_A, T), which is a model for many other dynamical systems. Let $g\colon \Sigma_A \to \mathbf{R}^+$. Let $g_n(x) = g(T^{n-1}x)$. If g is normalized in that

$$\sum_{y\in T^{-1}x} g(y) = 1 \quad (\forall x \in \Sigma_A),$$

then $\{g_n\}$ is a normalized sequence of potentials. The corresponding equilibrium states are the classical (homogeneous) equilibrium states. Note that if g is not normalized but of summable variation, it can be reduced to a normalized potential g_0.

The second special case is the Riesz product in harmonic analysis [**24**]. A general form of Riesz products is

$$\mu = \prod_{n=1}^{\infty}(1 + a_n\cos(2\pi\ell_1\cdots\ell_n x)), \quad (\ell \geqslant 3, |a_n| \leqslant 1\ \forall n).$$

The measure μ is understood to be the weak limit of the partial products of the above formal infinite product. Let $g_{n+1}(x) = \ell_{n+1}^{-1}(1 + a_n\cos(2\pi\ell_1\cdots\ell_n x))$ $(n \geqslant 0, \ell_0 = 1)$. If we identify $x \in [0,1)$ to the sequence $(x_n) \in X = \prod_{n=1}^{\infty}\{1,2,\ldots,\ell_n\}$ by

$$x = \sum_{n=1}^{\infty} \frac{x_n}{\ell_1 \ell_2 \cdots \ell_n},$$

then $\{g_n\}$ may be regarded as a normalized sequence of potentials on X. The Riesz product is the corresponding equilibrium state (if it is unique). A larger class is the set of G-measures studied by Brown and Dooley [**4**], which correspond to the full symbolic space defined by $S_n = \mathbf{Z}/\ell_n\mathbf{Z}$.

Return to the general case. Our first result concerns the existence and the uniqueness of equilibrium states.

THEOREM 1. *Let $G = \{g_n\}_{n\geqslant 1}$ be a normalized sequence of potentials defined on a transitive symbolic space X_A.*

(*a*) *The set of all equilibrium states associated to G is a non-empty convex compact set.*

(*b*) *There is a unique equilibrium state if and only if for any $f \in C(X_A)$, $P_n f(x)$ converges uniformly* (*in x*) *to a constant as $n \to \infty$.*

(*c*) *There is a unique equilibrium state if*

$$g_{\min} := \inf\{g_n(x) : x \in X_A, \quad n \geqslant 1\} > 0,$$

$$V := \sup\left\{\frac{G_n(x)}{G_n(y)} : n \geqslant 1; \quad x_j = y_j \quad (1 \leqslant j \leqslant n)\right\} < \infty.$$

(*d*) *Under the condition in* (*c*), *there exist constants D_1 and D_2 such that*

$$D_1 G_n(x) \leqslant \mu(I_n(x)) \leqslant D_2 G_n(x)$$

for all $x \in X_A$ and all $n \geqslant 1$, where $I_n(x) = \{y : y_j = x_j \; \forall 1 \leqslant j \leqslant n\}$.

We shall see that we may take $D_1 = g_{\min}^M / V$ and $D_2 = V$ where M is the least integer in the definition of transitivity of X_A.

Actually the constant in (*b*) is just $\int f d\mu$ where μ is the unique equilibrium state. Suppose without loss of generality that $\int f d\mu = 0$. Then $P_n f$ converges to zero as $n \to \infty$. Our main concern is the study of the convergence speed of $P_N f$. For $n \geqslant 1$ and any f, define

$$\mathrm{var}_n(f) = \sup\{|f(x) - f(y)| : x_j = y_j \quad (1 \leqslant j \leqslant n)\}.$$

For $1 \leqslant n \leqslant m$, let

$$V_{n,m} = -1 + \sup\left\{\frac{G_n(x)}{G_n(y)} : x_j = y_j \quad (1 \leqslant j \leqslant m)\right\}.$$

Note that $V_{n,m} \to 0$ as $m - n \to \infty$ if g_ns are rather regular.

THEOREM 2. *Let $G = \{g_n\}_{n\geqslant 1}$ be a normalized sequence of potentials defined on a transitive symbolic space X_A. There exists a constant $0 < \gamma < 1$ such that for any $f \in C(X_A)$, any $N \geqslant 1$ and any choice $1 \leqslant n_0 < n_1 < \cdots < n_k \leqslant N$, we have*

$$\|P_N f\|_\infty \leqslant \mathrm{var}_{n_0}(f) + \|f\|_\infty \left[\sum_{j=0}^{k-1} V_{n_j, n_{j+1}} + \gamma^k\right].$$

We shall see that we may take

$$\gamma = \frac{D_2 - D_1}{D_2 + D_1} = \frac{V^2 - g^M_{\min}}{V^2 + g^M_{\min}}.$$

These results are announced in [**11**]. The proof of Theorem 1 is based on ideas contained in [**6**, **9**] and the idea was used in [**10**]. The essential observation (Proposition 5) is the following projection property $P_n P_m = P_m P_n = P_m$ $(m \geqslant n)$. The proof of Theorem 2 is based on the following fact which is easy to prove (Lemma 2). Let E_n be the operator of conditional expectation with respect to the σ-field generated by the cylinders of length n. Then $E_n P_n$ is a strict contraction on the subspace of $L^p(\mu)$ consisting of functions with zero mean.

Now let us apply the above result to study the asymptotic behaviour of powers of transfer operators defined on a subshift of finite type. Recall that the *transfer operator* $L_g\colon C(\Sigma_A) \to C(\Sigma_A)$ associated to a non-negative function g is defined by

$$L_g f(x) = \sum_{y \in T^{-1}x} g(y) f(y).$$

where $T\colon \Sigma_A \to \Sigma_A$ is the subshift. If $g > 0$ and $\log g$ is of summable variation, i.e. $\sum_{n=1}^{\infty} \mathrm{var}_n(\log g) < \infty$ (equivalently $\sum_{n=1}^{\infty} \mathrm{var}_n(g) < \infty$), it is well known that the spectral radius ρ of L_g is an eigenvalue and there is a positive eigenfunction h associated to this eigenvalue. Consequently the study of L_g can be reduced to that of L_{g_0} where $g_0 = g + \log h - \log h \circ T - \log \rho$. In the sequel, assume that g is normalized and of summable variation. Let μ be the unique equilibrium state μ for g [**9**, **22**]. For a function $f\colon \Sigma_A \to \mathbf{R}$, the *correlation function* of f (or of the process $f(T^n x)$) is defined by

$$R_f(n) = \int f \circ T^n \cdot f d\mu - \left(\int f d\mu \right)^2, \quad (n \geqslant 1).$$

Since μ is mixing [**2**], $R_f(n)$ tends to zero for any continuous function f. In view of this, we may regard $R_f(n)$ as a rate of mixing. Note that if $\int f d\mu = 0$, we have

$$R_f(n) \leqslant \|f\|_\infty \|L_g^n f\|_1 \leqslant \|f\|_\infty \|L_g^n f\|_\infty.$$

THEOREM 3. *Let g be a normalized potential of summable variation defined on a transitive subshift of finite type. Let L_g be the corresponding transfer operator. There are constants $A > 0$ and $0 < \gamma < 1$ such that for any f with $\int f d\mu = 0$ (where μ is an equilibrium state), we have $R_f(N) \leqslant \|f\|_\infty \|L_g^N f\|_1$ and*

$$\|L_g^N f\|_\infty \leqslant A \left(\mathrm{var}_{n_0}(f) + \sum_{i=1}^{k} \sum_{j=n_{i+1}-n_i+1}^{n_{i+1}} \mathrm{var}_j(\log g) + \gamma^k \right)$$

for any $1 \leqslant n_0 < n_1 < \cdots < n_k \leqslant N$.

More precise estimates will be presented in Section 3 for special cases involving different kinds of regularities of f and g. We should note that our estimate on the supremum norm $\|L_g^N f\|_\infty$ is much stronger than the existing estimate on $R_f(N)$ or on the L^1-norm $\|L_g^N f\|_1$.

Theorem 2 also allows us to study the almost everywhere convergence of some lacunary series with respect to the non-homogeneous equilibrium state μ. Let $\{\alpha_n\}$ be a sequence of complex numbers and let $\{f_n\}$ be a sequence of μ-integrable functions on X_A. Suppose that $f_n(x)$ depends only upon $x_{n+1}, x_{n+2}, \ldots$. We would like to find conditions for the μ-almost everywhere convergence of the following series

$$\sum_{n=1}^{\infty} \alpha_n \left[f_n(x) - \int f_n(\cdot) d\mu(\cdot) \right].$$

THEOREM 4. *Let $\{g_n\}_{n\geqslant 1}$ be a normalized sequence of potentials defined on a transitive symbolic space and let $\{f_n\}$ be a sequence of functions such that $f_n(x)$ depend only upon $x_{n+1}, x_{n+2}, \ldots$. Suppose there are constants $A > 0$ and $\epsilon > 0$ such that for $m > n \geqslant 1$,*

$$\|f_n\|_\infty \leqslant A, \quad \mathrm{var}_m(f_n) \leqslant \frac{A}{(m-n)^{1+\epsilon}}, \quad \mathrm{var}_m(\log g_n) \leqslant \frac{A}{(m-n)^{2+\epsilon}}.$$

If $\sum_{n=1}^{\infty} |\alpha_n|^2 \log^2 n < \infty$, then the above series converges μ-almost everywhere.

Even for Riesz products, there were only very partial results about this convergence problem [**18**], which were obtained usually under the condition of analyticity of f_n (regarded as functions on the circle). Only for the case $f_n(x) = e^{2\pi i \ell_1 \cdots \ell_n x}$, was it proved that $\sum_n |\alpha_n|^2 < \infty$ is a necessary and sufficient condition for the almost-everywhere convergence [**7**, **19**]. Theorem 4 above requires only weaker conditions like Hölder continuity of f_n.

As a consequence, we get the following formula which gives the Hausdorff dimension of the non-homogeneous equilibrium state

$$\dim \mu = - \limsup_{n\to\infty} \frac{1}{\log(\ell_1 \ell_2 \cdots \ell_n)} \sum_{j=1}^{n} \int \log g_j(x) d\mu(x).$$

2. *Existence and uniqueness of equilibrium states*

The aim of this section is to prove Theorem 1. For convenience, we shall use the following notation. For $n \geqslant 1$, let

$$X_n = \prod_{j=1}^{n} S_j, \quad X^n = \prod_{j=n+1}^{\infty} S_j.$$

For a finite sequence x in X_n and an infinite sequence y in X^n, xy is used to denote the concatenation of x and y, which is a point in X. For $x = (x_j)_{j\geqslant 1} \in X$, $x|_n$ and $x|^n$ are used to denote the finite sequence $(x_1 x_2 \cdots x_n) \in X_n$ (the head of x) and the infinite sequence $(x_{n+1} x_{n+2} \cdots) \in X^n$ (the tail of x). On the restricted symbolic space X_A, we introduce

$$\Gamma_n = \{y \in X_n : y = x|_n \quad \text{for some } x \in X_A\}$$
$$\Gamma^n = \{y \in X^n : y = x|^n \quad \text{for some } x \in X_A\}.$$

Notice that the concatenation xy of a sequence $x \in \Gamma_n$ and a sequence $y \in \Gamma^n$ is a sequence in X_A if and only if $A_n(x_n, y_{n+1}) = 1$. For a fixed $x \in X_A$, let

$$\Gamma_n(x) = \{y \in X_n : y x|^n \in X_A\}.$$

It is clear that $\Gamma_n(x) \subset \Gamma_n$ and that $\Gamma_n(x) = \Gamma_n(y)$ if and only if $x_{n+1} = y_{n+1}$. It is worth noticing that Card $\Gamma_n = \|A_1 A_2 \cdots A_{n-1}\|_1$ for $n \geqslant 2$.

Recall that a sub-basis of the topology of X_A is the set of the cylinders (of lengths $n \geqslant 1$)

$$I(y) = \{x \in X_A : x|_n = y\} \quad (y \in \Gamma_n).$$

Note that $g_n(y_n x|^n)$ is not defined if $A_n(y_n, x_{n+1}) = 0$. We use the convention that $g_n(y_n x|^n) = 0$ when it is not defined, i.e. when $A_n(y_n, x_{n+1}) = 0$. With this convention, the sum defining the normalization condition of g_n in (2) can be taken over S_n.

Proposition 5. *Let $P_n : C(X_A) \to C(X_A)$ $(n \geqslant 1)$ be the averaging operators defined by* (3). *We have*

(i) $P_n 1 = 1$, $P_n^2 = P_n$.

(ii) $P_n P_m = P_m P_n = P_m$ *if* $1 \leqslant n \leqslant m$.

Proof. We prove $P_n 1 = 1$ by induction on n. It is obvious when $n = 1$. For $x \in X_A$, since $\Gamma_n(x)$ depends only on x_{n+1}, we write it as $\Gamma_n(x_{n+1})$. Now consider the decomposition

$$\Gamma_n(x_{n+1}) = \bigcup_{y_n : A_n(y_n, x_{n+1}) = 1} \Gamma_{n-1}(y_n) y_n,$$

where $\Gamma_{n-1}(y_n) y_n$ means $\{z y_n : z \in \Gamma_{n-1}(y_n)\}$. This decomposition allows us to write

$$\begin{aligned} P_n 1(x) &= \sum_{y_n : A_n(y_n, x_{n+1}) = 1} g_n(y_n x|^n) \sum_{y_1 \cdots y_{n-1} \in \Gamma_{n-1}(y_n)} G_{n-1}(y_1 \cdots y_n x|^n) \\ &= \sum_{y_n : A_n(y_n, x_{n+1}) = 1} g_n(y_n x|^n) = 1, \end{aligned}$$

where the reason for the second equality is that the inner sum equals 1 by induction. It is easy to see that $P_n f(x)$ is independent of $x|_n$ just because $\Gamma_n(x') = \Gamma_n(x'')$ if $x'|^n = x''|^n$. So, $P_n^2 = P_n$ and $P_n P_m = P_m$ when $n \leqslant m$. Now we have only to show $P_m P_n = P_m$ when $n < m$, which is actually equivalent to $\operatorname{Ker} P_n \subset \operatorname{Ker} P_m$. Suppose $P_n f(x) = 0$ for all $x \in X_A$. Then $P_m f(x)$ is equal to

$$\sum_{y_{n+1}, \cdots, y_m} \prod_{j=n+1}^{m} g_j(y_j \cdots y_m x|^m) \sum_{y_1 \cdots y_n} G_n(y_1 \cdots y_n y_{n+1} \cdots y_m x|^m) \times f(y_1 \cdots y_n y_{n+1} \cdots y_m x|^m),$$

where the condition on $(y_1, \cdots, y_m)$ is clear. However if the inner sum is 0, then $P_m f(x) = 0$.

Proof of Theorem 1. After establishing the proposition above, we enter into a general situation discussed in [**6**], from which we immediately get (a) and (b) of Theorem 1. (c) is a consequence of (d). In fact, let $\mathscr{B}^n$ be the σ-field generated by the image of P_n. It is easy to see that P_n may be regarded as the conditional expectation:

$$P_n f = \mathbf{E}_\mu(f | \mathscr{B}^n) \quad \mu\text{−a.e.}$$

for any equilibrium state. Let $\mathscr{B}^\infty$ be the limit of $\mathscr{B}^n$. An equilibrium state is said to be *ergodic* if it is trivial on $\mathscr{B}^\infty$. It is known in [**6**] that an equilibrium state is ergodic if and only if it is an extremal point in the convex set of all equilibrium states and that two ergodic equilibrium states are either mutually singular or identical. Since the conclusion of (d) implies that two equilibrium states are absolutely continuous with each other, (c) is then implied by (d). To prove (d), it suffices to follow the proof of proposition 3 in [**9**, p. 1245]. Let us give a sketch. Let μ be an equilibrium state. For any $n \geqslant 1$ and $M \geqslant 0$, we have

$$\mu(I(x|_n)) = \mathbf{E}_\mu[\mathbf{E}_\mu(\mathbf{1}_{I(x|_n)}|\mathscr{B}^{n+M})] = \mathbf{E}_\mu P_{n+M}\mathbf{1}_{I(x|_n)}(y).$$

The transitivity implies that there is an $M \geqslant 0$ such that for any $n \geqslant 1$, any $x_n \in S_n$ and any y_{n+M+1} there are $y^*_{n+1}, y^*_{n+2}, \dots, y^*_{n+M}$ verifying

$$A_n(x_n, y^*_{n+1})A_{n+1}(y^*_{n+1}, y^*_{n+2}) \cdots A_{n+M}(y^*_{n+M}, y_{n+M+1}) > 0.$$

It follows that

$$P_{n+M}\mathbf{1}_{I(x|_n)}(y) \geqslant G_{n+M}(x_1 \cdots x_n y^*_{n+1} y^*_{n+2} \cdots y^*_{n+M} y_{n+M+1} \cdots).$$

According to the conditions, we know that the right-hand side is bounded from below by $V^{-1}g^M_{\min}G_n(x)$. For the inverse inequality $\mu(I(x|_n)) \leqslant VG_n(x)$, it suffices to take $M = 0$.

3. *Estimation of* $\|P_n f\|_\infty$

In this section, we prove Theorem 2. The proof consists of several lemmas. One of the key results is the following elementary inequality.

Lemma 1. *Let* $0 < a < b < \infty$ *be two constants. There exists a constant* $0 < \gamma = \gamma(a, b) < 1$ *such that the inequality*

$$\left|\sum_{j=1}^n \alpha_j x_j\right| \leqslant \gamma \sum_{j=1}^n |\alpha_j| x_j$$

holds for any two sequences $\{\alpha_j\}$ *and* $\{x_j\}$ *of real numbers satisfying the following conditions*

$$\sum_{j=1}^n \alpha_j = 0, \quad a \leqslant x_j \leqslant b \ (1 \leqslant j \leqslant n)$$

($\gamma = (b - a)/(b + a)$ *is optimal*).

Proof. Suppose first $n = 2$. Since $\alpha_1 = -\alpha_2$, it suffices to show

$$|x_1 - x_2| \leqslant \gamma(x_1 + x_2).$$

We can assume $x_1 \leqslant x_2$. Let $y = x_2/x_1$. The inequality becomes $y - 1 \leqslant \gamma(y + 1)$, i.e. $y \leqslant (1 + \gamma)/(1 - \gamma)$. Note that $1 \leqslant y \leqslant b/a$. Choose γ such that $b/a = (1 + \gamma)/(1 - \gamma)$. Thus the inequality is proved in this special case. In the general case, let

$$y_1 = \sum_{j:\,\alpha_j \geqslant 0} \alpha_j x_j, \quad y_2 = \sum_{j:\,\alpha_j \leqslant 0} |\alpha_j| x_j.$$

Since the two sides of the desired inequality are homogeneous in α_js, we can assume

$$\sum_{j:\,\alpha_j \geqslant 0} \alpha_j = \sum_{j:\,\alpha_j \leqslant 0} |\alpha_j| = 1.$$

Note that $a \leqslant y_1, y_2 \leqslant b$. Then by the proved inequality for $n = 2$, we get

$$\left|\sum_{j=1}^{n} \alpha_j x_j\right| = |y_1 - y_2| \leqslant \gamma(y_1 + y_2) = \gamma \sum_{j=1}^{n} |\alpha_j| x_j.$$

Let $\mathscr{B}_n$ be the σ-algebra on X_A generated by the cylinders of length n. Let $E_n = \mathbf{E}(\cdot|\mathscr{B}_n)$ be the conditional expectation with respect to $\mathscr{B}_n$ on the probability space (X_A, μ).

LEMMA 2. *There exists a constant $0 < \gamma < 1$ such that for any $f \in L^\infty(\mu)$ with $\int f(x) d\mu(x) = 0$, we have*

$$\|P_n E_n f\|_\infty \leqslant \gamma \|f\|_\infty$$

(γ *is the one defined after the statement of Theorem* 2).

Proof. Note that

$$P_n E_n f(x) = \sum_{c \in \Gamma_n} G_n(cx|^n) \frac{1}{\mu(I(c))} \int_{I(c)} f d\mu$$

(where $G_n(cx|^n) = 0$ if $cx|^n \notin X_A$, by convention). Note also that $V^{-1} g_{\min}^M \leqslant (G_n(cx|^n))/(\mu(I(c))) \leqslant V$ (Theorem 1(d)) and $\sum_c \int_{I(c)} f d\mu = 0$. Applying the above lemma yields

$$\begin{aligned} |P_n E_n f(x)| &\leqslant \gamma \sum_{c \in \Gamma_n} G_n(cx|^n) \frac{\int_{I(c)} |f| d\mu}{\mu(I(c))} \\ &\leqslant \gamma \|f\|_\infty \sum_{c \in \Gamma_n} G_n(cx|^n) \\ &= \gamma \|f\|_\infty. \end{aligned}$$

It is also true that $\|P_n E_n f\|_1 \leqslant \gamma \|f\|_1$. For this, we integrate the above first inequality and use the fact that $\mu(I(c)) = \int_{X_A} G_n(cx|^n) d\mu(x)$ (see [**9**, **10**] for this fact). The Riesz–Thorin theorem assures then $\|P_n E_n f\|_p \leqslant \gamma \|f\|_p$ for any $1 \leqslant p \leqslant \infty$. These inequalities will not be used.

LEMMA 3. *If $m > n$, we have*

$$\mathrm{var}_m(P_n f) \leqslant \|f\|_\infty V_{n,m} + \mathrm{var}_m(f).$$

In particular, if f is $\mathscr{B}_n$-measurable, we have

$$\mathrm{var}_m(P_n f) \leqslant \|f\|_\infty V_{n,m}.$$

Proof. Suppose $x'|_m = x''|_m$. Then $\Gamma_n(x'_{n+1}) = \Gamma_n(x''_{n+1})$ and

$$\begin{aligned} &P_n f(x') - P_n(x'') \\ &\quad = \sum_{c \in \Gamma_n} G_n(cx'|^n) f(cx'|^n) - \sum_{c \in \Gamma_n} G_n(cx''|^n) f(cx''|^n) \\ &\quad = \sum_{c \in \Gamma_n} \left[G_n(cx'|^n) - G_n(cx''|^n)\right] f(cx'|^n) + \sum_{c \in \Gamma_n} G_n(cx''|^n) \left[f(cx'|^n) - f(cx''|^n)\right]. \end{aligned}$$

The absolute value of the last sum is bounded by

$$\mathrm{var}_m(f) \sum_{c\in\Gamma_n} G_n(cx''|^n) = \mathrm{var}_m(f).$$

The absolute value of the next to the last sum is bounded by

$$\|f\|_\infty \sum_{c\in\Gamma_n} G_n(cx''|^n) \left| \frac{G_n(cx'|^n)}{G_n(cx''|^n)} - 1 \right|.$$

Let $a = G_n(cx'|^n)$, $b = G_n(cx''|^n)$ and $\tau = V_{n,m} + 1$. By the definition of $V_{n,m}$, we have $\tau^{-1} \leqslant a/b \leqslant \tau$. If $a \geqslant b$, then

$$\left| \frac{a}{b} - 1 \right| = \frac{a}{b} - 1 \leqslant \tau - 1 = V_{n,m}.$$

If $a \leqslant b$, then

$$\left| \frac{a}{b} - 1 \right| = 1 - \frac{a}{b} \leqslant 1 - \tau^{-1} \leqslant \frac{\tau - 1}{\tau} < \tau - 1 = V_{n,m}.$$

It follows that

$$\sum_{c\in\Gamma_n} G_n(cx''|^n) \left| \frac{G_n(cx'|^n)}{G_n(cx''|^n)} - 1 \right| \leqslant V_{n,m} \sum_{c\in\Gamma_n} G_n(cx''|^n) = V_{n,m}.$$

The following lemma is obvious.

LEMMA 4. *For any f, we have*

$$\|(I - E_n)f\|_\infty \leqslant \mathrm{var}_n(f).$$

Proof of Theorem 2. Let $Q_n = P_n E_n$. First we remark that if $N \geqslant n$,

$$P_N = P_N[(I - E_n) + Q_n].$$

In fact, since $N \geqslant n$, we have $P_N = P_N P_n$ (Proposition 5). Then

$$P_N = P_N P_n (I - E_n) + P_N P_n E_n = P_N(I - E_n) + P_N Q_n.$$

By induction, we have

$$P_N = P_N \left[(I - E_{n_0}) + \sum_{j=1}^{k-1} (I - E_{n_j}) \prod_{i=0}^{j-1} Q_{n_i} + \prod_{i=0}^{k} Q_{n_i} \right].$$

By using the obvious fact that P_N is a contraction on $L^\infty(\mu)$, Lemmas 2–4, we have

$$\begin{aligned}
\|P_N f\|_\infty &\leqslant \|(I - E_{n_0})f\|_\infty + \sum_{j=1}^{k-1} \left\| (I - E_{n_j}) \prod_{i=0}^{j-1} Q_{n_i} f \right\|_\infty + \left\| \prod_{i=0}^{k} Q_{n_i} f \right\|_\infty \\
&\leqslant \mathrm{var}_{n_0}(f) + \sum_{j=1}^{k-1} \mathrm{var}_{n_j} \left(\prod_{i=0}^{j-1} Q_{n_i} f \right) + \gamma^k \|f\|_\infty \\
&\leqslant \mathrm{var}_{n_0}(f) + \|f\|_\infty \left[\sum_{j=1}^{k-1} V_{n_{j-1},n_j} + \gamma^k \right].
\end{aligned}$$

4. *Subshifts of finite type*

In this section, we shall deduce Theorem 3 from Theorem 2, then give some more precise estimates according to the regularity of the potential.

Proof of Theorem 3. Let $g_j(x) = g(T^{j-1}x)$. Let P_n be the corresponding averaging operators. First we have $|R_f(N)| \leqslant \|f\|_\infty \, \|L_g^N f\|_1$ which follows from the relation

$$R_f(N) = \int L_g^N(f \cdot f \circ T^N) d\mu = \int f \cdot L_g^N f d\mu.$$

Next observe that between L_g^n and P_n we have the relation $L_g^n f(T^n x) = P_n f(x)$. Since T is surjective, $\|L_g^N f\|_\infty = \|P_N f\|_\infty$. We are then led to estimate $\|P_n f\|_\infty$. Suppose $x'|_m = x''|_m$.

$$\begin{aligned} \frac{G_n(x')}{G_n(x'')} &\leqslant \exp \sum_{j=1}^{n} |\log g_j(x') - \log g_j(x'')| \\ &\leqslant \exp \sum_{j=1}^{n} \mathrm{var}_m(\log g_j) \\ &= \exp \sum_{j=1}^{n} \mathrm{var}_{m-(j-1)}(\log g) \\ &= \exp \sum_{j=m-n+1}^{m} \mathrm{var}_j(\log g). \end{aligned}$$

Since $e^x - 1 \leqslant Ax$ for $0 \leqslant x \leqslant B$, we have

$$V_{n,m} \leqslant A \sum_{j=m-n+1}^{m} \mathrm{var}_j(\log g).$$

If $\mathrm{var}_n(f) \leqslant A\alpha^n$, $\mathrm{var}_n(\log g) \leqslant A\beta^n$ for $n \geqslant 1$ ($A > 0, 0 < \alpha < 1, 0 < \beta < 1$ constants), then (cf. [**2**]) there exist $B = B(A, \alpha, \beta) > 0$ and $0 < \delta = \delta(A, \alpha, \beta) < 1$ such that

$$\|L_g^N f\|_\infty \leqslant B\delta^N \quad (N \geqslant 1).$$

This kind of convergence is said to be exponential. Now we see that we can get sub-exponential convergences when the variation of the potential decays to zero more slowly.

Theorem 6. *Suppose* $\int f d\mu = 0$ *and* $\|f\|_\infty = 1$.

(*a*) *If* $\mathrm{var}_n(f) \leqslant An^{-s}$, $\mathrm{var}_n(\log g) \leqslant An^{-r}$ *for* $n \geqslant 1$ ($A > 0, s > 0, r > 1$ *constants*), *then there exists* $B = B(A, s, t) > 0$ *such that*

$$\|L_g^N f\|_\infty \leqslant B \frac{(\log N)^{\max(s,t)}}{N^{\min(s,r-1)}} \quad (N \geqslant 1).$$

(*b*) *If* $\mathrm{var}_n(f) \leqslant A\alpha^{(\log n)^p}$, $\mathrm{var}_n(\log g) \leqslant A\alpha^{(\log n)^p}$ *for* $n \geqslant 1$ ($A > 0, 0 < \alpha < 1$, $p > 1$ *constants*), *then for any* $0 < \alpha' < \alpha$ *there exists* $B = B(A, \alpha', p) > 0$ *such that*

$$\|L_g^N f\|_\infty \leqslant B\alpha'^{(\log N)^p} \quad (N \geqslant 1).$$

(*c*) *If* $\mathrm{var}_n(f) \leqslant A\alpha^{n^\beta}$, $\mathrm{var}_n(\log g) \leqslant A\alpha^{n^\beta}$ *for* $n \geqslant 1$ ($A > 0, 0 < \alpha < 1, 0 < \beta < 1$ *constants*), *then for any* $0 < \alpha' < \alpha$ *there exists* $B = B(A, \alpha', \beta) > 0$ *such that*

$$\|L_g^N f\|_\infty \leqslant B\alpha'^{N^{\frac{\beta}{1+\beta}}} \qquad (N \geqslant 1).$$

Proof. In the sequel, C and C' etc are different constants in different places.

(*a*) Note that

$$\sum_{j=n+1}^{m} \frac{1}{j^s} \leqslant \frac{1}{s-1} \cdot \frac{1}{n^{s-1}}.$$

Apply Theorem 3 by choosing

$$n_0 = n_j - n_{j-1} = \left[\frac{N}{c \log N}\right] \quad (1 \leqslant j \leqslant k), \quad k = [c \log N] - 1,$$

where c is sufficiently large and $[a]$ denotes the integral part of a real number a. We get

$$\begin{aligned}\|L_g^N f\|_\infty &\leqslant C\left(\left(\frac{\log N}{N}\right)^\alpha + \log N\left(\frac{\log N}{N}\right)^{\beta-1} + \gamma^{c \log N}\right) \\ &\leqslant C' \frac{(\log N)^{\max(\alpha,\beta)}}{N^{\min(\alpha,\beta-1)}}.\end{aligned}$$

(*b*) Show first that for $n < m$, we have

$$\sum_{j=n+1}^{m} \alpha^{(\log j)^p} \leqslant \int_n^\infty \alpha^{(\log x)^p} dx \leqslant \frac{Cn}{(\log n)^{p-1}} \alpha^{(\log n)^p}.$$

In fact, using the inequality $(1+x)^p \geqslant 1+px$ ($p \geqslant 1, x \geqslant 0$), after a change of variable we get

$$\begin{aligned}\int_n^\infty \alpha^{(\log x)^p} dx &= n\alpha^{(\log n)^p} \int_1^\infty \alpha^{(\log y + \log n)^p - (\log n)^p} dy \\ &\leqslant n\alpha^{(\log n)^p} \int_1^\infty \alpha^{p(\log n)^{p-1} \log y} dy.\end{aligned}$$

The function under the last integral being a power of y, this integral is of size $(\log n)^{-(p-1)}$. Now to obtain the estimate for $\|L_g^n f\|_\infty$, it suffices to apply Theorem 3 by choosing

$$n_0 = n_j - n_{j-1} = \left[\frac{N}{\log^q N}\right] \quad (1 \leqslant j \leqslant k), \quad k = [\log^q N] - 1$$

where $q > p$ is some constant.

(*c*) As for (*a*) and (*b*), it suffices to show that

$$\int_n^\infty \alpha^{x^\beta} dx \leqslant C n^{1-\beta} \alpha^{n^\beta}$$

(the inverse inequality also holds for a smaller C). Observe that for $\lambda \geqslant 1$

$$\int_\lambda^\infty e^{-x^2} x dx = \tfrac{1}{2} e^{-\lambda^2}$$

$$\int_\lambda^\infty e^{-x^2} x^a dx = \tfrac{1}{2}\lambda^{a-1} e^{-\lambda^2} + \frac{a-1}{2} \int_\lambda^\infty e^{-x^2} x^{a-2} dx.$$

Let $a \geqslant 1$ and let q be the smallest integer such that $a - 2q \leqslant 1$. Applying q times the last equality enables us to get

$$\begin{aligned}\int_\lambda^\infty e^{-x^2} x^a dx &\leqslant C\left(\lambda^{a-1}e^{-\lambda^2} + \int_\lambda^\infty e^{-x^2} x^{a-2q} dx\right)\\ &\leqslant C\left(\lambda^{a-1}e^{-\lambda^2} + \int_\lambda^\infty e^{-x^2} x dx\right)\\ &\leqslant C'\lambda^{a-1}e^{-\lambda^2}.\end{aligned}$$

Now to obtain the inequality claimed at the beginning of the proof of (d), it suffices to apply the above inequality to the right-hand side of the following equality

$$\int_n^\infty \alpha^{x^\beta} dx = \frac{1}{|\log \alpha|^{1/\beta}} \int_{\sqrt{|\log \alpha| n^\beta}}^\infty e^{-z^2} z^{\frac{2}{\beta}-1} dz.$$

Apply Theorem 3 by choosing

$$n_0 = n_j - n_{j-1} = \left[N^{\frac{1}{1+\beta}}\right] \quad (1 \leqslant j \leqslant k), \quad k = \left[N^{\frac{\beta}{1+\beta}}\right] - 1.$$

Similar estimates for two-sided subshift of finite type can be deduced from the above results. Apply the estimate in Theorem 6(a) together with a result in [**16**], we get the following central limit theorem.

THEOREM 7. *Let g be a normalized potential defined on a transitive subshift of finite type such that* $\mathrm{var}_n(\log g) = O(n^{-2-\epsilon})$ *for some* $\epsilon > 0$. *Let f be a continuous function defined on the subshift space such that* $\mathrm{var}_n(f) = O(n^{-1-\epsilon})$. *Denote by μ the Gibbs measure associated to g. Then the limit*

$$\sigma = \lim_{n\to\infty} \frac{1}{\sqrt{n}} \left\| \sum_{j=0}^{n-1} f \circ T^j - n \int f d\mu \right\|_{L^2(\mu)}$$

exists and

$$\lim_{n\to\infty} \mu\left\{x: \sum_{j=0}^{n-1} f \circ T^j - n\int f d\mu \leqslant t\sqrt{n}\right\} = \frac{1}{\sqrt{2\pi}\sigma} \int_{-\infty}^t \exp\left(-\frac{x^2}{2\sigma^2}\right).$$

5. *Pointwise convergence of some series*

We shall prove the quasi-orthogonality of the series in Theorem 4. Then a variant of a Menchoff theorem [**14**] applies.

Proof of Theorem 4. Assume that $\int f_n d\mu = 0$ for all $n \geqslant 1$. Let $n < N \leqslant m$. By the invariance of μ, we have

$$\int f_n f_m d\mu = \int P_N(f_n f_m) d\mu = \int f_m P_N f_n d\mu.$$

Consequently,

$$\left|\int f_n f_m d\mu\right| \leqslant A \|P_N f_n\|_\infty.$$

As in the proof of Theorem 3, we can get

$$V_{n,m} \leqslant C \sum_{j=1}^{n} \mathrm{var}_m(g_j) \leqslant C' \sum_{j=1}^{n} \frac{1}{(m-j)^{2+\epsilon}} \leqslant C'' \frac{1}{(m-n)^{1+\epsilon}}.$$

Take $n < n_0 < n_1 < \cdots < n_k \leqslant N$. According to Theorem 2 and the hypothesis, we have

$$\|P_N f_n\|_\infty \leqslant \frac{A}{(n_0-n)^{1+\epsilon}} + A^2 \sum_{j=0}^{k-1} \frac{A}{(n_{j+1}-n_j)^{1+\epsilon}} + \gamma^k.$$

Choose $k = [a \log(m-n)] - 1$ with a sufficiently large so that $a \log 1/\gamma > 1$. Then choose n_js as follows

$$n_0 - n = n_1 - n_0 = \cdots = n_k - n_{k-1} = \left[\frac{m-n}{k+1}\right].$$

This is possible because $n_k - n \leqslant m - n$. Finally, choose $N = m$. Then, for some constant $C > 0$,

$$\begin{aligned}\|P_N f_n\|_\infty &\leqslant C\left[\left(\frac{\log(m-n)}{m-n}\right)^{1+\epsilon} + \log(m-n)\left(\frac{\log(m-n)}{m-n}\right)^{1+\epsilon} + \frac{1}{(m-n)^{a \log 1/\gamma}}\right] \\ &\leqslant \frac{C'}{(m-n)^{1+\epsilon'}}\end{aligned}$$

for some $0 < \epsilon' < \epsilon$. This implies that the system $\{f_n\}$ is quasi-orthogonal in $L^2(\mu)$, so the series converges according to a Menchoff theorem [**14**].

Let us consider G-measures [**4**] defined on the circle $\mathbf{T} = \mathbf{R}/\mathbf{Z}$, which include Riesz products. Let $G = \{g_n\}$ be a sequence of positive functions defined on the circle and ℓ_n a sequence of integers such that $\{\ell_n\} \geqslant 2$. Suppose for any $n \geqslant 1$

$$\sum_{j=1}^{\ell_n} g_n\left(\frac{j}{\ell_n} + x\right) = 1 \quad (\forall x \in \mathbf{T}).$$

The weak limit (if it exists) of the following infinite product is called a G-measure:

$$\mu = \prod_{n=1}^{\infty} [\ell_n g_n(\lambda_{n-1}x)] \quad (\lambda_0 = 1, \lambda_n = \ell_1 \cdots \ell_{n-1}).$$

To state the next result, we use $\omega(f,t) = \sup_{|x-y|\leqslant t} |f(x) - f(y)|$ to denote the modulus of continuity of a function f defined on the circle.

THEOREM 8. *Suppose for some $C > 0$ and $\epsilon > 0$ we have*

$$\omega(\log g_n, t) \leqslant \frac{C}{|\log^{2+\epsilon} t|}.$$

Then the above weak limit exists and defines a G-measure μ. Moreover, if we are given a sequence $\{f_n\}$ of continuous functions defined on the circle such that

$$\|f_n\|_\infty \leqslant C, \quad \omega(f_n, t) \leqslant \frac{C}{|\log^{1+\epsilon} t|}.$$

Then the following series

$$\sum_{n=1}^{\infty} \alpha_n \left[f_n(\lambda_{n-1}x) - \int f_n(\lambda_{n-1}\cdot) d\mu(\cdot) \right]$$

converges μ-almost everywhere whenever $\sum_{n=1}^{\infty} |\alpha_n|^2 \log^2 n < \infty$.

Proof. We can identify the circle $\mathbf{T}$ with the symbolic space $X = \prod_{n=1}^{\infty}\{1, 2, \dots, \ell_n\}$ by the following mapping

$$(x_n) \in X \longrightarrow \sum_{n=1}^{\infty} \frac{x_n}{\ell_1 \ell_2 \cdots \ell_n} \in \mathbf{T}.$$

The G-measure is just the equilibrium state (if it is unique) associated to the potentials defined by

$$g_n(x_n x_{n+1} \cdots) = g_n \left(\sum_{j=0}^{\infty} \frac{x_{n+j}}{\ell_n \cdots \ell_{n+j}} \right).$$

(We use the same letter g_n to denote two functions defined respectively on $\mathbf{T}$ and X. There should be no confusion.) The condition on g_n implies that

$$\mathrm{var}_m(g_n) \leqslant O\left(\frac{1}{|m-n|^{2+\epsilon}} \right) \quad (m \neq n).$$

This in turn implies that the constant V in Theorem 1(c) is finite. Then we have the uniqueness of the equilibrium state and the existence of G-measure. The condition on f_n implies the conditions of Theorem 4.

When $g_{n+1}(x) = \ell_{n+1}^{-1}[1+\mathrm{Re}\, a_n e^{2\pi i \lambda_n x}]$, we get a Riesz product. In this case, Peyrière proved the convergence when f_n are analytical functions [**18**]. The above theorem generalizes Peyrière's result to G-measures, improving the result by relaxing the analyticity to a weak condition on the modulus of continuity. However, Peyrière required only ℓ^2-summability for the sequence of coefficients $\{\alpha_n\}$. For the special case $f_n(x) = e^{2\pi i x}$ ($\forall n$), it was proved in [**7**, **19**] that the ℓ^2-summability is actually necessary and sufficient for the series to converge almost everywhere with respect to the Riesz product.

6. *Dimensions of equilibrium states*

By using Theorem 4, we will get a formula for the dimension of the equilibrium state. In general, for a Borel measure μ defined on X_A (or on any metric space), we define its upper dimension and lower dimension, respectively denoted by $\dim^* \mu$ and $\dim_* \mu$, as follows

$$\dim^* \mu = \inf\{\dim F \colon \mu(F^c) = 0\}$$

$$\dim_* \mu = \sup\{\alpha \geqslant 0 \colon \dim E < \alpha \Longrightarrow \mu(E) = 0\},$$

where $\dim B$ denotes the Hausdorff dimension of a Borel set B. When $\dim^* \mu = \dim_* \mu (= \alpha)$, we say that μ is unidimensional or α-dimensional and we write simply $\dim \mu = \alpha$ (see [**8**] for a general account).

THEOREM 9. *Suppose that there is a $\epsilon > 0$ such that*

$$\mathrm{var}_m(\log g_n) = O((m-n)^{-2-\epsilon}) \quad (\forall m > n \geqslant 1).$$

Then the unique equilibrium state μ is unidimensional and

$$\dim \mu = -\limsup_{n\to\infty} \frac{1}{\log(\ell_1\ell_2\cdots\ell_n)} \sum_{j=1}^{n} \int \log g_j(x)d\mu(x).$$

Proof. Define the local (lower) dimension of μ at x by

$$\underline{D}(\mu, x) = \liminf_{n\to\infty} \frac{\log \mu(I_n(x))}{\log |I_n(x)|},$$

where $|I_n(x)| = (\ell_1\ell_2\cdots\ell_n)^{-1}$. It is known in [**8**] that $\dim^* \mu$ (resp. $\dim_* \mu$) equals the essential upper bound (resp. lower bound) of $\underline{D}(\mu, x)$ with respect to μ. By Theorem 1(d), we have

$$\frac{\log \mu(I_n(x))}{\log |I_n(x)|} = -\frac{1}{\log(\ell_1\cdots\ell_n)} \sum_{j=1}^{n} \log g_j(x) + o(1).$$

With the Kronecker lemma in mind, we have only to show the almost-everywhere convergence of the following series

$$\sum_{n=1}^{\infty} \frac{1}{\log(\ell_1\cdots\ell_n)} \left[\log g_n(x) - \int \log g_n(x)d\mu(x)\right].$$

However, since $\log(\ell_1\cdots\ell_n) \geqslant n\log 2$, the convergence is assured by Theorem 4.

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