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Meshfree methods and boundary conditions

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SUMMARY

We perform here some meshfree methods to inhomogeneous Laplace equations. We prove the efficiency of those methods compared with classical ones, for one or two dimensional case for numerics, and for one dimensional for theorical results. Copyright © 2000 John Wiley & Sons, Ltd.

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1. Introduction

Consider a bounded connected domain Ω whose boundary $\partial\Omega$ is smooth. Consider a standard elliptic PDE problem on that domain. When looking for a finite element approximation for that problem, we begin with meshing the domain Ω. It is standard to observe that the approximation results depend on how accurate the mesh approximates the boundary ([15]).

In this article, we are concerned with non standard methods where the shape functions have not to vanish on $\partial\Omega$. Our study takes part into the meshfree methods (see [3] and the references therein).

Loosely speaking, let us explain on a simple exemple what is a meshfree method: let us pretend that we want to analyze the properties of some function whose support is included in Ω approximated by finite elements or wavelets. Either we construct some ad hoc wavelets or splines whose support is included in Ω, or we use the whole family of splines or wavelets defined on $\mathbb{R}^n$. The latest method is a meshfree method.

Let us now give an overview of usual meshfree methods applied to our problem. Consider a standard elliptic problem, where some Dirichlet condition is imposed on some part Γ_D of the boundary. When performing a Galerkin FEM approximation, we use test functions that vanish on Γ_D. This is no longer possible with meshfree methods, since we perform the computations

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pretending that we have no knowledge of the boundary, at least at first stage. To overcome this difficulty, we usually proceed as follows

(i) Either we modify the meshfree functions whose supports encounter the boundary in order to get new functions that vanish on that boundary. When dealing with wavelets, this has been performed in [7] when Ω is a one dimensional interval and in [18] for particular geometries in higher dimension. The drawback of the method is that it is very difficult to implement and that it depends on the gemetry of the domain.

(ii) Or we use fictious domain. Introducing $\tilde{\Omega}$ a suitable square that includes Ω, we use a suitable operator to extend functions defined on Ω to the square. Hence we solve the minimization problem (whose elliptic problem is the Euler equation) on functions defined on $\tilde{\Omega}$, taking into account the boundary conditions on $\partial\Omega$ as Lagrange multipliers for the minimization problem. We refer here to [1] and to [8,10] for the particular case of wavelets. Unfortunately, it turns out that this method is not as accurate as expected and expensive due to the use of Lagrange multipliers

Therefore, we need to introduce another way to proceed.

In this article we apply some meshfree method to a standard elliptic problem supplemented with inhomogeneous mixed boundary conditions. The idea is to incorporate the boundary conditions into a non standard variational formulation. This allows us to forget the boundary in our choice of approximation. Unfortunately, the drawback of the method is that we are led with a bilinear form that is no longer definite positive. The same phenomena occurs when dealing with discontinuous Galerkin methods (see [14] and the references therein).

The article is organized as follows: In Section 2, we discuss the non standard variational formulation and the equivalence with the original problem. In Section 3, we give a complete description of the corresponding 1D problem. Section 4 is devoted to some 2D results. Numerics take place in Section 5: in a first subsection we compare our meshfree approach to the classical FEM, then in a second subsection we describe some numerical results where Ω is a bounded polygonal domain of $\mathbb{R}^2$.

2. Non standard variational formulation

2.1. *The elliptic problem*

Consider a bounded connected domain Ω whose boundary $\partial\Omega$ is smooth. We assume throughout this article that the boundary of Ω splits into two parts, $\partial\Omega = \Gamma_D \cup \Gamma_N$, such that the $n-1$ dimensional measure of Γ_D is not 0. In that case, the following Poincaré inequality holds true (see [4]) : set $H^1_D(\Omega)$ for the subset of functions in $H^1(\Omega)$ whose trace vanishes on Γ_D. Then there exists a numerical constant c such that for all $u \in H^1_D(\Omega)$

$$||u||_{L^2(\Omega)} \leq c||\nabla u||_{L^2(\Omega)}. \quad (1)$$

Consider f, u_0, g to be specified later. We are looking for a u solution to

$$-\Delta u = f \text{ in } \Omega, \tag{2}$$

$$u = u_0 \text{ on } \Gamma_D, \tag{3}$$

$$\frac{\partial u}{\partial n} = g \text{ on } \Gamma_N. \tag{4}$$

We now make the following regularity assumption: we assume that f belongs to $L^2(\Omega)$, that u_0, g are the traces of some smooth functions and that satisfies some compatibility conditions on $\bar{\Gamma}_D \cap \bar{\Gamma}_N$ in order to ensure that (2) possesses a (unique) solution in $H^2(\Omega)$. For the precise statements on the data f, u_0, g we refer to [11]. Consider a test function v that belongs to $C^1(\bar{\Omega})$. Then (2) leads to, using the classical Stokes formula,

$$\int_\Omega \nabla u.\nabla v dx - \int_{\partial\Omega} \frac{\partial u}{\partial n} v d\sigma = \int_\Omega f v dx. \tag{5}$$

We now want to use

$$\int_{\partial\Omega} \frac{\partial u}{\partial n} v d\sigma = \int_{\Gamma_N} g v d\sigma + \int_{\Gamma_D} \frac{\partial u}{\partial n} v d\sigma. \tag{6}$$

Let us observe that this is not valid for v in $H^1(\Omega)$, since the restriction map from $L^2(\partial\Omega)$ to $L^2(\Gamma_D)$ does not extend to a map from $H^{1/2}(\partial\Omega)$ to $H^{1/2}(\Gamma_D)$. In other words, $\int_{\partial\Omega} \frac{\partial u}{\partial n}(\chi_{\Gamma_D} v) d\sigma$ does not makes sense in general, even if v belongs to $C^1(\bar{\Omega})$. We then make one of the following splitting assumptions

(i) either $\bar{\Gamma}_D \cap \bar{\Gamma}_N = \emptyset$. This holds true for instance if Ω is an annulus of $\mathbb{R}^2$ or an interval in 1D, or if Γ_N is empty.

(ii) or the trace of the functions u, u_0 and v of Γ_D belong to $H^{1/2}_{00}(\Gamma_D)$, wich is the space of functions such that, extended by 0 on Γ_N belong to $H^{1/2}(\partial\Omega)$ [11, 17]. The trace of u and v on Γ_N also have to belong to $H^{1/2}_{00}(\Gamma_N)$. In particular, u, v and u_0 have to vanish on $\bar{\Gamma}_D \cap \bar{\Gamma}_N$. We will see in the following that we can make numerics without this condition and we obtain although good results.

Remark

These conditions inforce regularity of r.h.s. of (2).

We now proceed to the computations assuming that (6) is valid for any test function v. For numerics, we are interested to deal with symmetric bilinear forms. For that purpose, we substitute to (5), (6) the following equation

$$\int_\Omega \nabla u.\nabla v dx - \int_{\Gamma_D} (\frac{\partial u}{\partial n} v + \frac{\partial v}{\partial n} u) d\sigma = \int_\Omega f v dx + \int_{\Gamma_N} g v d\sigma - \int_{\Gamma_D} \frac{\partial v}{\partial n} u_0 d\sigma. \tag{7}$$

Throughout this article, we set $\beta(u, v)$ for the l.h.s of (7) and we set $\Lambda(v)$ for the r.h.s of (7). The variational formulation holds true if (6) is valid for any v in $C^1(\bar{\Omega})$. It is then straightforward to establish

Proposition 1. *Assume that (6) is valid for any v in $C^1(\bar{\Omega})$. Then the elliptic problem (2) is equivalent to the variational formulation (7). In that case, the non standard variational formulation has a unique solution.*

Remarks.

(i) Even if (6) is not valid, then using (7) to compute approximations of the solution will provide us with interesting results; see Section 5 below.

(ii) The classical formulation to handle this inhomogeneous mixed problems reads as follows: find u such that $u - u_0$ belongs to $H^1_D(\Omega)$ such that

$$\int_\Omega \nabla u.\nabla v dx = \int_\Omega f v dx + \int_{\Gamma_N} g v d\sigma, \tag{8}$$

for all $v \in H^1_D(\Omega)$. The drawback of the method is that the knowledge of a lifting of u_0 is required. See Section 5 in [4].

(iii) Another classical approach is the mixed variational formulation (see [5]) that reads: find (u, q) in $H^1(\Omega) \times Q$ such that

$$\int_\Omega \nabla u.\nabla v dx - \int_{\Gamma_D} q v d\sigma = \int_\Omega f v dx + \int_{\Gamma_N} g v d\sigma, \tag{9}$$

$$\int_{\Gamma_D} p u d\sigma = \int_{\Gamma_D} p u_0 d\sigma, \tag{10}$$

for all (v, p) in $H^1(\Omega) \times Q$. Here q plays the role of $\frac{\partial u}{\partial n}$ and $Q = \{\eta \in H^{-1/2}(\partial\Omega), \eta = 0 \text{ on } \Gamma_D\}$.

2.2. Properties of the variational formulation

First of all, we observe that since $\beta(1, 1) = 0$, then the bilinear form is not definite positive. Nevertheless there exists a large subspace of $H^1(\Omega)$ where the restriction of β is definite positive: $H^1_D(\Omega)$. This is a mere consequence of (1).

On the other hand, it is easy to see that β and Λ are not continuous on $H^1(\Omega)$; nevertheless, the continuity on $H^2(\Omega)$ holds true. Then we will use later either test functions in $H^2(\Omega)$ or in $C^1(\bar{\Omega})$.

3. The one-dimensional case

3.1. The variational formulation

Without loss of generality, we may assume that $\Omega =]0, 1[$. We then have two different cases to handle: Dirichlet boundary conditions, where $\Gamma_D = \{0, 1\}$ and mixed boundary conditions where $\Gamma_D = \{0\}$. In the latest case, (7) reads: given f in $L^2(\Omega)$ and a, b in $\mathbb{R}$, find u such that for any v in $C^1(\bar{\Omega})$

$$\int_0^1 u'v'dx + (u'(0)v(0) + v'(0)u(0)) = \int_0^1 f v dx + bv(1) + av'(0). \tag{11}$$

In the first case, we then have: given f in $L^2(\Omega)$ and a, b in $\mathbb{R}$, find u such that for any v in $C^1(\bar{\Omega})$

$$\int_0^1 u'v'dx + (u'(0)v(0) + v'(0)u(0)) - (u'(1)v(1) + v'(1)u(1)) = \int_0^1 fvdx - bv'(1) + av'(0). \tag{12}$$

Remark. Let us observe that we do not need to consider C^1 functions as test functions for (12) or (11). Continuous, piecewise C^1 (up to the boundary) functions can also be used .

3.2. Discrete formulation

Consider $h > 0$ that plays the role of the mesh size. Consider some function ϕ whose support is compact. We assume that ϕ is continuous and piecewise C^1. Throughout this article, we shall refer to ϕ as the *scaling function.* Consider

$$\phi_k(x) = h^{-1/2}\phi(\frac{x}{h} - k), \text{ for } k \in \mathbb{Z}. \tag{13}$$

Let us observe that the width of the support of ϕ_k is $O(h)$. Consider E_h the finite dimensional subspace spanned by the functions ϕ_k whose support matches $\bar{\Omega}$. We assume that the functions ϕ_k are a uniform Riesz basis for the space they spanned, i.e. that there exists some numerical constant $c > 0$ such that for any sequence a_k

$$c\sum_{k\in\mathbb{Z}} a_k^2 \leq ||\sum a_k\phi_k||^2_{L^2(\mathbb{R})} \leq \frac{1}{c}\sum_{k\in\mathbb{Z}} a_k^2. \tag{14}$$

Remark. There are numerous examples of such functions. Consider for instance the hat function for P^1 finite elements $\phi(x) = \max(0, 1 - |x|)$ or any smooth enough compacted supported wavelet (see [12] and the references therein).

We suppose that $\cup_{h>0}E_h$ is dense in $H^1(\Omega)$. We also assume that E_h satisfies the classical Strang-Fix condition: 1 and x belong to E_h. This ensures that (see [16])

Proposition 2. *There exists a numerical constant C such that for any u in $H^2(\Omega)$*

$$\text{Inf}||u - u_h||_{L^2(\Omega)} + h\text{Inf}||u' - u_h'||_{L^2(\Omega)} \leq Ch^2||u||_{H^2(\Omega)}, \tag{15}$$

being understood that the infima are computed on E_h.

We now introduce the discrete formulation that reads as follows: find u_h in E_h such that

$$\beta(u_h, v_h) = \Lambda(v_h), \tag{16}$$

for all v_h in E_h. We now prove error estimates for these formulations in the next two subsections.

3.3. One dimensional inhomogeneous mixed boundary conditions case

We first state a result about the well-posedness of the discrete version of (11) that reads as follows

$$\int_0^1 u_h' v_h' dx + (u_h'(0)v_h(0) + v_h'(0)u_h(0)) = \int_0^1 f v_h dx + b v_h(1) + a v_h'(0), \tag{17}$$

for all $v_h \in E_h$.

Proposition 3. *For any small enough $h > 0$, the discrete version (17) possesses a unique solution u_h.*

Proof. Any function $v_h \in E_h$ splits into

$$v_h(x) = v_h(0)(1 - x) + w_h(x), \tag{18}$$

where w_h belongs to $V_h = E_h \cap H_D^1(\Omega)$, i.e the subspace of functions in E_h that vanish on the Dirichlet boundary part $\{0\}$. Introducing M the matrix of β/E_h (the matrix of the bilinear form β on the space E_h) , we then have that M reads

$$M = \begin{pmatrix} A & B^* \\ B & -1 \end{pmatrix} \tag{19}$$

where A is the symmetric definite positive matrix of β/V_h, where $-1 = \beta(1 - x, 1 - x)$, and where B is a one row and $\dim V_h$ columns matrix. We now use the following lemma that is standard when dealing with mixed finite elements (see [5]).

Lemma 1. *Consider A a symmetric definite positive $n \times n$ matrix, C a symmetric positive $p \times p$ matrix and B a $p \times m$ matrix. If*

$$M = \begin{pmatrix} A & B^* \\ B & -C \end{pmatrix}, \tag{20}$$

then a vector $\begin{pmatrix} u \\ q \end{pmatrix}$ belongs to $KerM$ if and only if $u = 0$ and $q \in KerC \cap KerB^*$.

Applying this lemma completes the proof of the proposition. □

We now state an error estimate that shows that our method compares with classical finite elements methods

Theorem 1. *There exists a numerical constant C such that if u (respectively u_h) denotes the solution of (11) (respectively (17)) then*

$$||u - u_h||_{L^2(\Omega)} + h||u' - u_h'||_{L^2(\Omega)} \leq Ch^2||f||_{L^2(\Omega)}. \tag{21}$$

Proof. Due to the very definition of u, u_h we know that

$$\beta(u - u_h, v_h) = 0, \forall v_h \in E_h. \tag{22}$$

We apply this formula to two functions whose trace does not vanish on the boundary, $v_h = 1$ and $v_h = x$ (and that belong to E_h thanks to the Strang-Fix condition). We then have

$$u_h'(0) = u'(0), \tag{23}$$

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$$u_h(1) = u(1). \tag{24}$$

This leads to

$$\beta(u - u_h, u - u_h) = \int_0^1 (u' - u_h')^2 dx. \tag{25}$$

We combine (25) and (22) to obtain that for any v_h in E_h

$$\begin{aligned} \int_0^1 (u' - u_h')^2 dx = \beta(u - u_h, u - v_h) = \\ \int_0^1 (u' - u_h')(u' - v_h')dx + (u(0) - u_h(0))(u'(0) - v_h'(0)). \end{aligned} \tag{26}$$

We now perform the change of unknowns $w(x) = u(x) - xu'(0)$ in (26) to obtain

$$\int_0^1 (u' - u_h')^2 dx = \int_0^1 (u' - u_h')(w' - v_h')dx - (u(0) - u_h(0))v_h'(0). \tag{27}$$

We thus obtain

$$||u' - u_h'||_{L^2(\Omega)} \leq \inf ||w' - v_h'||_{L^2(\Omega)}, \tag{28}$$

where the infimum is computed on the functions $v_h \in E_h$ that satisfy $v_h'(0) = 0$.

We now consider a suitable choice of v_h that is constructed as follows. Since the scaling function ϕ is compactly supported, there exists a finite number m of functions ϕ_k (see (13)) whose support contains $\{0\}$; moreover m is independent of h. We set $k \sim 0$ if the support of ϕ_k contains $\{0\}$.

On the other hand, due to the Strang-Fix condition, x reads as a series

$$x = \sum_k \gamma_k \phi(x - k). \tag{29}$$

This series converges on any compact subset of $\mathbb{R}$. We now define an ansatz of 1 defined in a neighborhood of $\{0\}$ as follows

$$1_0 = \sum_{k\sim 0} \gamma_k \phi'(\frac{x}{h} - k). \tag{30}$$

Let us observe that $1_0 = 1$ on a neighborhood of 0, that 1_0 belongs to $E_h' = \{v_h'; v_h \in E_h\}$. We also have the straightforward estimate: there exists a numerical constant C such that

$$||1_0||_{L^\infty(\Omega)} + h^{-1/2}||1_0||_{L^2(\Omega)} \leq C. \tag{31}$$

We now proceed to the computations. We set $w' = w_h' + r$ where w_h' belongs to E_h' and where r satisfies

$$\int_0^1 r v_h' dx = 0, \forall v_h \in E_h. \tag{32}$$

This means that w_h' is the L^2 projection of w' on E_h'. Therefore, due to (15), we have that

$$||r||_{L^2(\Omega)} \leq Ch||w''||_{L^2(\Omega)} = Ch||f||_{L^2(\Omega)}. \tag{33}$$

We now use Pythagore theorem to get

$$||w' - v'_h||^2_{L^2(\Omega)} = ||r||^2_{L^2(\Omega)} + ||w'_h - v'_h||^2_{L^2(\Omega)}. \tag{34}$$

At this stage, we specify $v'_h(x) = w'_h(x) - w'_h(0)1_0(x)$, that belongs to E'_h and that satisfies $v'_h(0) = 0$. For this choice of v'_h, we then have

$$||w'_h - v'_h||_{L^2(\Omega)} = |w'_h(0)|||1_0||_{L^2(\Omega)} \leq Ch^{1/2}|w'_h(0) - w'(0)|, \tag{35}$$

due to (31) and to $w'(0) = 0$. Using classical approximation result as (15), we then have

$$|w'_h(0) - w'(0)| \leq ||w'_h - w'||_{L^\infty(\Omega)} \leq Ch^{1/2}||w''||_{L^2(\Omega)}. \tag{36}$$

This leads to the following L^2 estimate on the gradient of u

$$||u'_h - u'||_{L^2(\Omega)} \leq Ch||f||_{L^2(\Omega)}. \tag{37}$$

We now proceed to the L^2 estimate using a classical Aubin-Nitsche argument (see [6]). Consider a test function T that satisfies $||T||_{L^2(\Omega)} = 1$. We now prove that there exists a numerical constant C such that

$$\int_\Omega T(u - u_h)dx \leq Ch^2||f||_{L^2(\Omega)}, \tag{38}$$

and the proof of the theorem will be completed.

To begin with, let us introduce the linear form

$$\Lambda_T(v) = \int_\Omega Tvdx + av'(0) + bv(1). \tag{39}$$

Then, we solve

$$\beta(\psi, v) = \Lambda_T(v), \forall v \in C^1([0, 1]), \tag{40}$$

$$\beta(\psi_h, v_h) = \Lambda_T(v_h), \forall v \in E_h. \tag{41}$$

Using (37), we then have

$$||\psi'_h - \psi'||_{L^2(\Omega)} \leq Ch||T||_{L^2(\Omega)} = Ch. \tag{42}$$

On the other hand, using (22), (23), (24), we obtain

$$\int_\Omega T(u - u_h) = \beta(\psi, u - u_h) = \beta(\psi - \psi_h, u - u_h). \tag{43}$$

Since $\psi'(0) - \psi'(0) = u'(0) - u'_h(0) = 0$, then

$$\int_\Omega T(u - u_h) = \int_\Omega (\psi' - \psi'_h)(u' - u'_h)dx, \tag{44}$$

and Cauchy-Schwarz inequality together with (37) completes the proof of the Theorem. □

3.4. Inhomogeneous Dirichlet boundary conditions

We first state a result about the well-posedness of the discrete version of (12) that reads as follows

$$\int_0^1 u'_h v'_h dx + (u'_h(0)v_h(0) + v'_h(0)u_h(0)) - (u'_h(1)v_h(1) + v'_h(1)u_h(1)) = \int_0^1 f v_h dx - bv'_h(1) + av'_h(0), \tag{45}$$

for all $v_h \in E_h$.

Proposition 4. *For any small enough $h > 0$, the discrete version (45) possesses a unique solution u_h.*

Proof. Any function v_h in E_h reads

$$v_h = v_h(0)(1-x) + v_h(1)x + w_h(x), \tag{46}$$

where w_h belongs to $V_h = H^1_D(\Omega) \cap E_h$.

Therefore the matrix M of β reads

$$M = \begin{pmatrix} A & B^* \\ B & -C \end{pmatrix}, \tag{47}$$

where A is the symmetric definite positive matrix of β/V_h, where B is a two rows and $\dim V_h$ columns matrix. $-C$ is the matrix of β restricted to the space spanned by $(1, x)$ and reads

$$C = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}. \tag{48}$$

We now use Lemma 3.3: if a vector $\begin{pmatrix} u \\ q \end{pmatrix}$ belongs to Ker$M$ if and only if $u = 0$ and $q \in$ Ker$C \cap$ KerB^*. In our case, $q \in$ Ker$C$ implies that q is colinear to 1. $q \in$ Ker$B^* - \{0\}$ implies that $\beta(1, v_h) = 0 = v'_h(0) - v'_h(1)$ for any v_h in E_h. This is straightforward to construct v_h that does not satisfies this equality. Then the proof of the Proposition is completed. □

We now state

Theorem 2. *There exists a numerical constant C such that if u (respectively u_h) denotes the solution to (12) (respectively (45)) then*

$$||u - u_h||_{L^2(\Omega)} + h||u' - u'_h||_{L^2(\Omega)} \leq Ch^2 ||f||_{L^2(\Omega)}. \tag{49}$$

Proof. We first have

$$\beta(u - u_h, 1 - x) = u'(0) - u'_h(0) = 0, \tag{50}$$

$$\beta(u - u_h, x) = u'(1) - u'_h(1) = 0. \tag{51}$$

Using now (45), (50) we obtain

$$\int_0^1 (u' - u_h')^2 dx = \beta(u - u_h, u - u_h) = \beta(u - u_h, u - v_h), \tag{52}$$

for all $v_h \in E_h$. Then, if $v_h' = u'$ on $\partial\Omega$,

$$\int_0^1 (u' - u_h')^2 dx = \int_0^1 (u' - u_h')(u' - v_h') dx, \tag{53}$$

and therefore

$$\int_0^1 (u' - u_h')^2 dx = \inf \int_0^1 (u' - v_h')^2 dx, \tag{54}$$

where the infimum is computed onto the space of functions in E_h that satisfy $v_h' = u'$ on $\partial\Omega$. We now proceed as in the previous subsection. Let us introduce two ansatz functions 1_0 and 1_1 defined in a analogous way than in (30). Introducing the L^2 orthogonal projection w_h' of u' into the space $E_h' = \{v_h'; v_h \in E_h\}$, we now consider $v_h'(x) = w_h'(x) + (u'(0) - w_h'(0))1_0(x) + (u'(1) - w_h'(1))1_1(x)$ and thus obtain (due to (31))

$$\inf \int_0^1 (v' - v_h')^2 dx \leq ||u' - w_h'||_{L^2(\Omega)} + Ch^{1/2}||u' - w_h'||_{L^\infty(\Omega)}. \tag{55}$$

We now use classical approximation results to get the L^2 estimate on the gradient of the error. To prove the estimate L^2 we proceed as above by an Aubin-Nitsche argument. Since the proof is very similar, we omit it. □

4. The two-dimensional problem

4.1. The discrete problem

Consider E_h a finite dimensional space that is spanned by functions $\{\Phi_k(x) = h^{-1}\Phi(\frac{x}{h} - k); k \in \mathbb{Z}^2\}$, where we only consider functions Φ_k whose support matches $\bar{\Omega}$. We may suppose that we are considering a scaling function ϕ that reads $\Phi(x_1, x_2) = \phi(x_1)\phi(x_2)$ where ϕ is the one-dimensional scaling function.

We plan to solve: $\forall v_h \in E_h$,

$$\beta(u_h, v_h) = \Lambda(v_h). \tag{56}$$

A first difficulty appears: we do not know if (56) is well-posed, i.e. if the underlined linear system has a unique solution. The only fact we know is that if we consider the space $\dot{E}_h$ of the functions compactly supported in Ω, then the restriction of β to that space provides us with a symmetric definite positive matrix. The complementary of $\dot{E}_h$ into E_h has dimension $O(h^{-1})$ when $\dim(E_h) = O(h^{-2})$.

Remarks.

(i) Generically, the underlined matrix is inversible. This is the case in the numerical examples of the next section.

(ii) In the particular case $\Gamma_D = \partial\Omega$, J.A. Nitsche [13] gives a non conforming formulation where the finite dimensional space V_h is a subspace of $H^2(\Omega)$ and

$$\beta_h(u_h, v_h) = \beta(u_h, v_h) + \kappa_h \int_{\Gamma_D} u_h v_h \; d\sigma \tag{57}$$

where the penalisation parameter κ_h is properly chosen such that β_h becomes definite positive. In particular, κ_h tends to inifinity when h tends to zero. The convergence of u_h to the solution u is also obtained.

(iii) Using the discontinous Galerkin method, ones rather consider the non symetric penalized bilinear form

$$\tilde{\beta}_h(u_h, v_h) = \int_\Omega \nabla u_h \cdot \nabla v_h \; dx - \int_{\Gamma_D} (\frac{\partial u}{\partial n} v - u \frac{\partial v}{\partial n} \; d\sigma) + \kappa_h \int_{\Gamma_D} u_h v_h \; d\sigma. \tag{58}$$

The avantage of this formulation is that κ_h no longer needs to tends to $+\infty$ when h tends to zero (see [14] and references therein).

4.2. A remark about the continuous problem

Going back to (2), one can decompose the solution u as $u = v + w$ where

$$-\Delta v = -\Delta u \text{ in } \Omega \tag{59}$$

$$v = 0 \text{ on } \Gamma_D \tag{60}$$

$$\frac{\partial v}{\partial n} = \frac{\partial u}{\partial n} \text{ on } \Gamma_N. \tag{61}$$

and

$$-\Delta w = 0 \text{ in } \Omega \tag{62}$$

$$w = u \text{ on } \Gamma_D \tag{63}$$

$$\frac{\partial w}{\partial n} = 0 \text{ on } \Gamma_N. \tag{64}$$

We already know that $\beta(v, v) = \int_\Omega |\nabla v|^2 dx$ since $v \in H^1_D(\Omega)$. Moreover, we have

Lemma 2.

$$\beta(w, w) = -\int_\Omega |\nabla w|^2 dx \le 0 \tag{65}$$

Proof. Due to the Stokes formula

$$\begin{aligned} \beta(w, w) = \int_\Omega |\nabla w|^2 dx - 2 \int_{\Gamma_D} w \frac{\partial w}{\partial n} d\sigma = \\ -\int_\Omega w \Delta w + \int_{\Gamma_N} w \frac{\partial w}{\partial n} d\sigma - \int_{\Gamma_D} w \frac{\partial w}{\partial n} d\sigma, \end{aligned} \tag{66}$$

that completes the proof of the lemma since $\frac{\partial w}{\partial n} = 0$ on Γ_N and $\Delta w = 0$ in Ω. □

5. Numerical results

The purpose of this section is to perform numerical computations in order to validate the method.

5.1. One-dimensional results

We first present computations on the one-dimensional problem

$$\begin{cases} -u''(x) = f(x) & if\ x \in (0,1) \\ u(0) = a; u'(1) = b \end{cases} \tag{67}$$

We consider this problem with different test functions :

- P1 finite elements, with a standard formulation and with the new method described above;
- P2 finite elements, with a standard formulation and with the new method described above;
- Daubechies compactly supported wavelets [9] of order 3, with a support included in the interval [0,5];
- B-Spline wavelets [7] of order 3, wich is a convolution between two P1 finite elements.

Throughout the remaining of this section the figures plot the L^2-error (or H^1-error) in logarithm scale versus the step of discretization (the size of wavelets) in logarithm scale.

Remark. When we use wavelets, due to the scale relation

$$\phi(x) = \sum_{k\in[0,M]} h_k \phi(2x - k) \tag{68}$$

where M is a finite integer depending on the order of the wavelet (or equivalently of the width of the support), the coefficient of the discretized stiffness matrix

$$K_{k\ell} = \int_I \phi'_{jk}(x)\phi_{j\ell}(x)dx \tag{69}$$

where $\phi_{jk}(x) = 2^{j/2}\phi(2^j x - k)$, can be computed exactly when I is an dyadic interval, without any quadrature formula[2, 10]. This is a very important property of wavelets. When I is not dyadic, a Gauss quadrature formula is used.

5.1.1. Checking the consistency In this paragraph, we show the consistency of the method when wavelets are used (Daubechies wavelet of order 3 and B-Spline wavelet of order 3). These fonctions can reproduce exactly polynoms of degree less or equal to 2. With a standard variational formulation, consistency is then also equal to 2. The patch test have been passed when the solution u is equal to 1, x, and x^2, except in one case, with an error less than 10^{-12}.

We have chosen to present here, in (67) the following r.h.s.:

$$f(x) = -2;\ a = 1;\ b = 2;$$

such that the solution is the polynom $u(x) = x^2 + 1$. The results in table I are obtained with 8 wavelets on interval (0,1), there are similar when u is any polynom of degree less than 2.

We perform the computation has been made in two cases:

- First, we have discretized the interval (0,1) only with dyadic sub-intervals, such that the computation of the stiffness matrix can be performed exactly without any quadrature formula (see remark above). We refer to this as the *dyadic* case.
- Then, the discretization of (0,1) is such that intersections of the supports of the wavelets and (0,1) are not always dyadic. In this case, a standard Gauss quadrature formula is used to compute the corresponding coefficients of the discretized stiffness matrix. We refer to this as the *non dyadic* case.

Table I. Numerical L2-consistency

Wavelet	L^2 error	H^1 error
B-Spline, dyadic	$1.09\ 10^{-12}$	$3.78\ 10^{-12}$
B-Spline, non dyadic	$9.82\ 10^{-13}$	$3.31\ 10^{-12}$
Daubechies, dyadic	$9.62\ 10^{-12}$	$3.51\ 10^{-9}$
Daubechies, non dyadic	$3.42\ 10^{-3}$	$1.21\ 10^{-1}$

We obtain a numerical consistency conform to the theory, except in the non dyadic case for Daubechies wavelets. The reason why is the non regularity of this function, giving an important error in the quadrature formula used for the computation of the stiffness matrix.

5.1.2. Spectral analysis of operator Let (70) be the eigenvalue problem

$$\begin{cases} -\Delta u(x) = \lambda u(x) & if\ x \in \Omega \\ u(x) = 0 & if\ x\ on\ \Gamma_D \\ \frac{\partial u}{\partial n} = 0 & if\ x\ on\ \Gamma_N \end{cases} \tag{70}$$

The eigenvalues for this problem are denoted by $\lambda_i,\ i \geq 1$, with $\lambda_1 < \lambda_2, ...$

Using a standard Galerkin method, we have the proposition [15]

Proposition 5. *Let a be the continuous, symetric and V-elliptic bilinear form associated to the classical variational formulation of (70)*

$$a(u,v) = \int_\Omega \nabla u \nabla v dx = \lambda \int_\Omega uv dx \qquad \forall v \in H^1_D(\Omega).$$

Consider E_h the finite dimensional subspace of V satisfying: there exists an integer $k \geq 1$ such as for all integer ℓ with $1 \leq \ell \leq k$ and for all $v \in H^{\ell+1}(\Omega)$:

$$\inf\{\|v - v_h\|_{L^2(\Omega)} + h\|\nabla v - \nabla v_h\|_{L^2(\Omega)}, v_h \in E_h\} \leq Ch^{\ell+1}\|v\|_{H^{\ell+1}(\Omega)}, \tag{71}$$

where C is a positive constant, independent of V_h.

Then the eigenvalues of the discretized problem

$$a(u_h, v_h) = \lambda \int_\Omega u_h v_h dx \qquad \forall v_h \in E_h$$

form an non decreasing sequence $\lambda_{1,h} \leq \lambda_{2,h} \leq ... \leq \lambda_{I,h}$.

If the subset V_m of the first corresponding eigenvectors satisfies $V_m \subset H^{\ell+1}(\Omega)$ for an integer ℓ with $1 \leq \ell \leq k$, then

$$|\lambda_{m,h} - \lambda_m| \leq Ch^{2\ell}. \tag{72}$$

We now perform some numerics on the non standard formulation, in the case of the one dimensional problem :

$$\begin{cases} -u''(x) = \lambda u(x) & if\ x \in (0,1) \\ u(0) = u'(1) = 0 \end{cases} \tag{73}$$

Figure 1 plots the relative error (in logarithmic scale) $\frac{|\lambda_5 - \lambda_{5,h}|}{|\lambda_5|}$ versus the step h of discretization (in logarithmic scale).

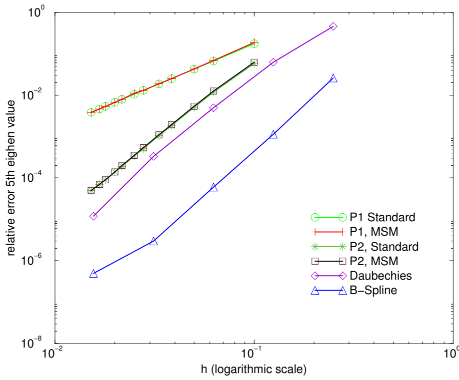

Figure 1. Relative error for the 5^{th} eigenvalue

Convergence rate is given in table II.

Table II. L^2 convergence rate of the 5^{th} eigenvalue of Laplacian operator

method	observed α	theorical α
P1, standard	2.02	2
P1, non standard	2.04	2
P2, standard	3.81	4
P2, non standard	3.85	4
Daubechies	3.80	4
B-Splines	3.99	4

We can observe that we obtain the same result both with standard and non standard formulation, and the theorical and observed rate are very close.

5.2. *Two-dimensional results*

The purpose of this paragraph is to present numerical computations performed in two dimensions to study if the theorical results proved in one dimension are still valid in higher dimensions.

We have chosen to solve the problem (2) on the following three different geometries

1. A dyadic square whose edges are located on lines $x = \frac{1}{2^p}$ or $y = \frac{1}{2^p}$, where p is an integer. Hence the coefficients of the stiffness matrix can be computed exactly without any quadrature formula (see Remark 68).
2. A square where coordinates of faces are not dyadic. Here, a quadrature formula is needed to compute coefficients of the stiffness matrix. This test allows us to evaluate the influence of the approximation in the integral computations.
3. A pentagon wich is the first step towards general polygonal domains.

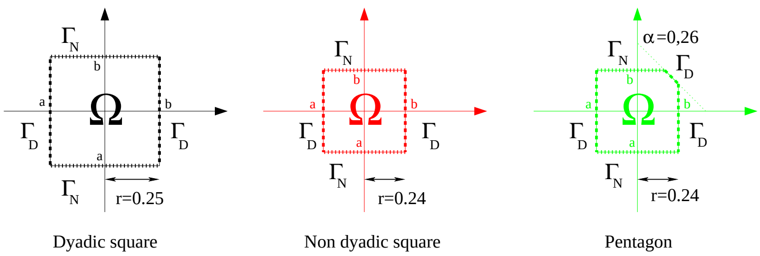

Figure 2. Geometries

All computations presented in this section are made with B-Spline wavelet of order 3.

5.2.1. Checking the consistency The first step to estimate the accuracy of a method is to compute its consistency. We present the result obtained with B-Splines of order 3, that reproduce polynomial functions of order less than 2 exactly. With standard formulation, consistency is then also equal to 2.

Figure 3 plots L^2-error in logarithm scale versus the step of discretization in logarithm scale (the size of wavelets). Data for the Dirichlet problems (2) are the following

$$\left\{\begin{array}{ll} f(x,y) = -4 & if\ (x,y) \in \Omega \\ g(x,y) = 2y & if\ (x,y)\ on\ \Gamma_N \\ u_0(x,y) = x^2 + y^2 & if\ (x,y)\ on\ \Gamma_D \end{array}\right. \tag{74}$$

The solution is equal to $u(x,y) = x^2 + y^2$ for (x,y) in Ω.

The consistency for order 2 is also obtained with our formulation (numericaly, with an L^2-error less than 10^{-11}), on the three different geometries. The results are the same when the solution u is any polynom of x and y of degree less than 2, such as 1, x, y, xy, x^2 and y^2.

5.2.2. Speed rate of convergence We now test if the speed of convergence proved in the one-dimensional case (see theorem 1 and 2) is still valid in 2D, as long as numerical computations are concerned.

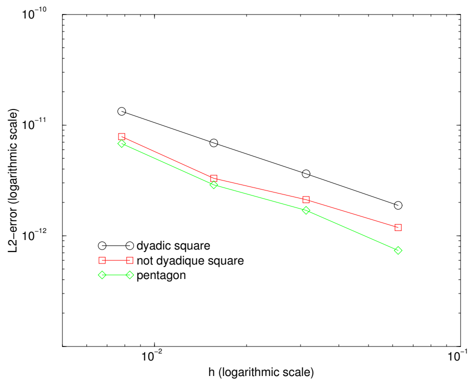

Figure 3. Verification of consistency

With smooth function When the consistency is of order n, we can expect a L^2 convergence rate as $O(h^\alpha)$ with $\alpha = n+1$.

Figure 4 plots this convergence rate when the data are

$$\begin{cases} f(x,y) = 2\cos x \sin y & if\ (x,y) \in \Omega \\ g(x,y) = \cos x \sin y & if\ (x,y)\ on\ \Gamma_N \\ u_0(x,y) = \cos x \sin y & if\ (x,y)\ on\ \Gamma_D \end{cases} \tag{75}$$

and the solution is equal to $u(x,y) = \cos x \sin y$ in Ω. According to different geometries, the L^2 convergence rate are the following

Table III. L^2 convergence rate for different geometries

geometries	α
dyadic square	3.0842
not dyadic square	2.8598
pentagon	2.8666

and the theorical convergence rate, using standard formulation, is equal to $\alpha = 3$. Then, the non standard formulation compares with the classical ones.

Non smooth functions with boundary layer In this paragraph, we study what happens when the external force f is not regular. More precisely, we take $f(x,y) = -(x-a)^{-1/2+\varepsilon}$ if $(x,y) \in \Omega$, where ε is a small parameter, and a is such that $x = a$ on the left boundary of Ω (wich is in Γ_D). The Neumann condition g is 0 on Γ_N, and the Dirichlet condition u_0 is equal to $C(x-a)^{3/2+\varepsilon}$ where $C = \frac{4}{(1+2\varepsilon)(3+2\varepsilon)}$. The solution u of the problem (2) is equal to $u(x,y) = C(x-a)^{3/2+\varepsilon}$ when (x,y) is in Ω.

In this case, the theorical L^2 convergence rate is equal to $\alpha = 2+\varepsilon$ with B-Spline wavelets of order 3. The observed L^2 convergence rate, presented in table IV and in figure 5 shows that

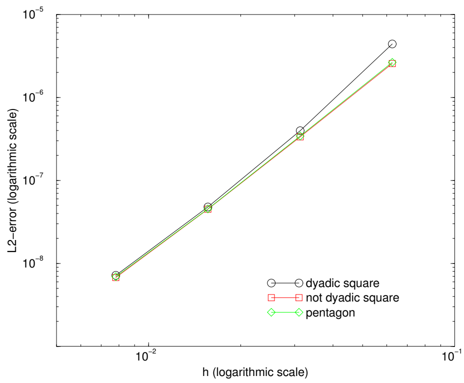

Figure 4. Smooth function

results close to the expectations.

Table IV. L^2 convergence with B-Spline wavelets of order 3 on a pentagon and f not regular

ε	observed α	theorical α
0.10	2.1272	2.10
0.05	1.9782	2.05
0.01	1.9385	2.01

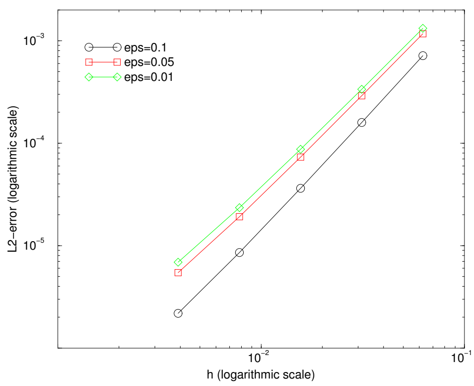

Figure 5. Non smooth functions with boundary layer

6. Conclusion

In this article, a non standard variational formulation is presented to take into account Dirichlet boundary conditions directly in the variational formulation and not in the space of test functions. We perform the full theorical study of this approach is made in the one-dimensional case.

Numerical results are presented in dimension one and two. The efficiency of these results compares to those of a standard formulation, and allows us to easily use test functions such as wavelets, even on a general domain on $\mathbb{R}^2$.

This technique is easy to compute, and differs from the classical formulation only near from the boundary. Inside the domain, if the shape functions are wavelets for example, all numerical integrations can be rapidly computed. The complexity of the calculus is the proportional to the size of the boundary.

In consequence, it becomes more efficient to use this technique than adapt wavelets in order that they vanish on the boundary of the domain.

For all these reasons, we still develop this method, in dimension greater than one for theorical aspects, and to numerically solve dynamical problems.

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