Dynamical Systems in Mathematical Physics

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1. Introduction

The purpose of this article is to describe some basic problems related to the interplay between Dynamical Systems and Mathematical Physics. Since it is impossible to be exhaustive in these topics, we choose to focus on water waves models. These mathematical models are described by partial differential equations that can be understood as dynamical systems in a suitable infinite-dimensional phase space.

We will not address the original equations for 2D surface water waves, even if we know that dynamical systems methods can help to exhibit some solitary waves for the equations; we refer here to the works of Iooss, Kirchgarsner, Mielke. Another approach is to seek these 2D surface water waves as saddle points for some Hamiltonian energies; see Buffoni, Séré, Tolland.

We rather present these arguments on some asymptotical models for the propagation of surface water waves.

2. Asymptotical models in hydrodynamics

To begin with, consider an irrotational fluid in a canal that is governed by the Euler equations and that is subject to gravitational forces. If the canal has finite depth, Boussinesq, 1877, and Korteweg-de Vries, 1890, have obtained the following model for unidirectional long waves

$$u_t + u_x + u_{xxx} + uu_x = 0. \qquad (2.1)$$

Sometimes we drop the u_x term in the left-hand side of (2.1), thanks to a suitable change of coodinates. Alternatively, we can also deal with the so-called generalized KdV equation that reads

$$u_t + u_{xxx} + u^k u_x = 0. \qquad (2.2)$$

where k is some positive integer. There are also other models designed to represent long waves in shallow water. Let us introduce the regularized long-wave equation (also referred as the Benjamin-Bona-Mahony equation) that reads

$$u_t - u_{txx} + u_x + uu_x = 0, \qquad (2.3)$$

or the Camassa-Holm equation

$$u_t - u_{txx} + 3uu_x = 2u_x u_{xx} + uu_{xxx}. \qquad (2.4)$$

For deep water, a well-known model was introduced by Zakharov, 1968,

$$iu_t + u_{xx} + \varepsilon |u|^2 u = 0, \qquad (2.5)$$

and describes the slow modulations of wave packets. Here the unknown $u(x,t)$ takes values in $\mathbb{C}$, and this nonlinear Schrödinger equation is in fact a system. In these equations, ε is either 1 or -1; throughout this article we shall refer to the former case as the focusing case and to the latter as the defocusing case. We also may substitute $|u|^{2p}u$ to the nonlinear term in (2.5) to get alternate models.

The variable t represents the time and the space variable x belongs either to $\mathbb{R}$ or to a finite interval when we are dealing with periodic flows.

The models above are intended to describe the propagation of unidirectional waves. For two-way waves, we refer to the article by Bona, Chen and Saut, 2002.

Actually, these equations feature particular solutions, the so-called travelling waves. Let us recall for instance that for generalized KdV equation (2.2) these solutions are

$$u(t,x) = Q_c(x-ct), \tag{2.6}$$
$$Q_c(x) = c^{\frac{1}{p}} Q(\sqrt{c}x), \tag{2.7}$$
$$Q(x) = (3\mathrm{ch}^{-2}(px))^{\frac{1}{p}}. \tag{2.8}$$

Fig. 1. Soliton

These so-called *solitons*, move to the right without changing their shape; c is the speed of propagation. In real life, this phenomena was observed by Russel, 1834. Riding his horse, he was able to follow for miles the propagation of such a wave on the canal from Edinburgh to Glasgow. On the other hand, Camassa-Holm equations are designed to describe the propagation of *peaked* solitons as

Fig. 2. peaked soliton

Focusing nonlinear Schrödinger equations also feature solitary waves that read $u_\omega(t,x) = \exp(i\omega t)Q(x)$, where Q is solution to

$$Q_{xx} - \omega Q + Q^{2p+1} = 0; \tag{2.9}$$

There are numerous examples of equations or systems of equations that model 2D-surface water waves. Among all these models, a first issue is to figure which one are relevant as long as dynamical properties are concerned. Indeed, we address here the question of stability of solitary waves (up to the symmetries of the equation). For instance, the *orbital stability* for cubic Schrödinger reads: For any $\varepsilon > 0$, there exists a neighborhood Ω of $u_\omega(x,0)$ such that any trajectory starting from Ω satisfies

$$\sup_t \inf_\theta \inf_y ||u(t) - \exp(i\theta)u_\omega(t,.-y)||_{H^1} \leq \varepsilon. \tag{2.10}$$

Another issue consists in the interaction of N-solitons. Schneider and Wayne, 2000, have addressed the issue of the validity of water waves models when this interaction is concerned.

Assume now that the validity of these models is granted. To consider (2.1) or (2.5) as a dynamical system, the next issue is then to consider the initial value problem.

3. The initial value problem

Let us supplement these equations with initial data u_0 in some Sobolev space. We shall consider either

$$H^s(\mathbb{R}) = \{u; \int_{\mathbb{R}} (1+|\xi|^2)^s |\hat{u}(\xi)|^2 d\xi < +\infty\}, \quad (3.1)$$

in the case where x belongs to the whole line, or the corresponding Sobolev space when we are considering periodic boundary conditions. We wonder if these equations provide a continuous flow $S(t) : u_0 \to u(t)$ in these functional spaces (at least locally in time). We would like to point out that for each Sobolev space under consideration, we may have a different flow. This fact is at the heart of the area of infinite dimensional dynamical systems.

The initial value problem was a challenge for decades for low norms, that is for small s. The last breakthrough was performed by Bourgain, 1993. Let us present the method for Korteweg-de Vries equation. Consider $U(t)u_0$ the solution of the Airy equation

$$u_t + u_{xxx} = 0; u(0) = u_0. \quad (3.2)$$

Without going into further details, the idea is to perform a fixed point argument to the Duhamel's form of the equation,

$$u(t) = U(t)u_0 - \frac{1}{2}\int_0^t U(t-s)\partial_x(u^2(s))ds, \quad (3.3)$$

in a suitable mixed space-time Banach space whose norm reads $||U(-t)u(t,x)||_{H_t^b H_x^\rho}$. This feat of skill relies on fine properties in harmonic analysis. Thanks to this method, we know that the Schrödinger equation (2.5) and the KdV (2.1) are well-posed respectively in $H^s(\mathbb{R}), s \geq 0$ and in $H^s(\mathbb{R}), s > -\frac{3}{4}$, locally in time. For the periodic case, the results are slightly different. We would like to point out that both KdV and nonlinear Schrödinger equations provide semi-groups $S(t)$ that do not feature smoothing effect. A trajectory that starts from H^s remains in H^s; indeed, we can also solve these partial differential equations backward in time.

The next issue is to figure out if these flows are defined for all times. Loosely speaking, the following alternative holds true: either the local flow in H^s extends to a global one, or some *blow up* phenomena occurs, that is $||S(t)u_0||_{H^s}$ collapses in finite time.

To this end, let us observe that, for instance, the mass $\int_{\mathbb{R}} |u(x)|^2 dx$ is conserved for both Korteweg-de Vries and nonlinear Schrödinger flows. Therefore one can prove that the solutions in L^2 are global in time. It is worthwhile to observe that the Bourgain method provides also some global existence results *below* the energy norm

Consider now the flow of the solutions in H^1. The second invariant for nonlinear Schrödinger equations reads

$$\int_{\mathbb{R}} |u_x(x)|^2 dx - \frac{\varepsilon}{p+1}|u(x)|^{2p+2}dx. \quad (3.4)$$

Therefore, the local solutions in H^1 extend to global ones in the defocusing case ($\varepsilon = -1$). In the focusing case, the situation is more contrasted. The solution is global if the nonlinearity is less than a H^1-critical value ($p = 2$ for Schrödinger and $k = 4$ for generalized Korteweg-de Vriese-quation). This critical value depends on some Sobolev embedings as

$$\int_{\mathbb{R}} |u(x)|^{2p+2} dx \leq C_p ||u||_{L^2}^{p+1} ||u_x||_{L^2}^p. \quad (3.5)$$

Therefore, since the mass is constant, the second invariant controls the H^1 norm of the solution if $p < 2$. Let us observe that the critical power of the nonlinearity depends also on the dimension of the space; it is the cubic Schrödinger that is critical in $H^1(\mathbb{R}^2)$. It is well-known that for some initial data, blow up phenomena can occur for two-dimensional cubic Schrödinger equations. Moreover, the behavior of blow up solutions is more or less understood. This analysis was performed using the *conformal invariance* of the equation. For quintic Schrödinger equation, which is critical in one dimension, this conformal invariance states that if $u(t,x)$ is solution, then

$$v(t,x) = |t|^{-\frac{1}{2}}\exp(\frac{ix^2}{4t})\bar{u}\left(\frac{1}{t},\frac{x}{t}\right) \quad (3.6)$$

is also solution.

On the other hand, for the generalized KdV equation, there is no conformal invariance and the blow up issue was open for years. There were some numerical evidences that blow up can occur for $k = 4$. Recently, Martel and Merle, 2002, have given a complete description of the blow up profile for this equation. Their methods are involved and rely on an ejection of mass at infinity in a suitable coordinates system.

We observe that in our discussion, we have presented some quantities that are invariant by the flow of the solutions. This is related to the Hamiltonian structure of the dynamical systems under consideration.

4. Hamiltonian systems in hydrodynamics

The study of Hamiltonian systems has developed beyond celestial mechanics (the famous n-body problems) to other fields in Mathematical Physics. We focus here on dynamical systems that read

$$u_t = J\frac{\partial}{\partial u}H(u), \quad (4.1)$$

where H is the Hamiltonian and J some skew-symmetric operator. For instance (2.1) is an Hamiltonian system with $J = \partial_x$ (that is an unbounded skew-symmetric operator) and

$$H(u) = \frac{1}{2}\int(u^2 - u_x^2)dx + \frac{1}{6}\int u^3 dx. \quad (4.2)$$

There is a subclass of Hamiltonian systems that are integrable by inverse scattering methods. For instance, (2.1) belongs to this class. Indeed, these methods give a complete description of the asymptotics when t goes to $+/-\infty$. It is well known (Deift, Zhou 1993) that, asymptotically, any solution to KdV consists of a wave train moving to the right in the physical space up to a dispersive part moving to the left.

On the other hand, a generic Hamiltonian system is not integrable. The study of the asymptotics and of the dynamical properties of such a system deserves another analysis. We say that a system features *asymptotic completness* if there exists u_+ ans u_- such that the solution $u(t)$ of (4.1) supplemented with initial data u_0 satisfies

$$||u(t) - U(t)u_+|| \to 0, \quad (4.3)$$

$$||u(t) - U(t)u_-|| \to 0 \quad (4.4)$$

when, respectively, $t \to +\infty$ or $t \to -\infty$. Here $U(t)u_0$ is the solution of the *free* equation, that is, the associated linear equation, supplemented with initial data u_0; for instance, the Airy equation is the free equation related to Korteweg-de Vries equation. The operators $u_- \rightarrow u_0 \rightarrow u_+$ are called *wave operators*. This is related to the Bohr's transition in Quantum Mechanics. Loosely speaking, we are able to prove these *scattering* properties for high powers in the nonlinearity for subcritical defocusing Schrödinger equations.

The asymptotics of trajectories can be more complicated. Let us recall that the stability of travelling waves is also an important issue to understand the dynamical properties of these models. For instance, let us point out that Martel and Merle proved the asymptotic stability of the sum of N-solitons for KdV in the subcritical case.

Beyond these asymptotics we are interested in the case where the permanent regime is chaotic (or turbulent). A scenario is that there exists quasi-periodic solutions of arbitrarly order N for the system under consideration. The next challenge about these Hamiltonian systems is to apply the Kolmogorov-Arnold-Moser theory to exhibit this kind of solutions to systems like (4.1). Here we restrict our discussion to the case of bounded domains, with either periodic or homogeneous Dirichlet conditions. Then, let us introduce the following definition: a solution is quasi-periodic if there exists a finite number N of frequencies ω_k such that

$$u(t,x) = \sum_{l=1}^{N} u_l(x) \exp(i\omega_k t). \qquad (4.5)$$

This extends the case of periodic solutions ($N = 1$), that are isomorphic to the torus. To prove the existence of such structures one idea is then to imbed N-dimensional invariant tori into the phase space of solutions. One may pretend approximate the infinite-dimensional Hamiltonian by a sequence of finite ones and consider the convergence of iterated simplectic transformations, or one solves directly some nonlinear functional equation. Actually, the difficulty is that resonances can occur. Resonances occur when there are some linear combination of the frequencies that vanish (or that is arbitrarely close to 0). This introduces a small divisor problem in a phase space that has *infinite dimension* To overcome these difficulties a Nash-Moser scheme can be implemented (see Craig, 1996). There are numerous open problems in that directions. For instance, let us observe that known results are essentially in the case where the dimension of the ambiant space is one. On the other hand, quasi-periodic solutions correspond to N-dimensional invariant tori for the flow of solutions; one may seek for Lagrangian invariant tori that correspond to the case where $N = +\infty$. Current research is directed towards extending this analysis.

Another issue is to seek invariant measures for these Hamitonian dynamical systems, as in Statistical Mechanics. Bourgain was successfull when performing this analysis for some nonlinear Schrödinger equations either in the case of periodic boundary conditions or in the whole space. This result is an important step in the ergodic analysis of our Hamitonian dynamical systems. This could explain the Poincaré recurrence phenomena observed numerically for these kind of equations: some particular solutions seem to come back to their initial state after a transient time. This point will not be developed here.

All these results are properties of *conservative* dynamical systems. We now address the case when some dissipation takes place.

5. Dissipative water waves models

To model the effect of viscosity on 2D surface water waves, we go back to a flow governed by the Navier-Stokes equations and we proceed to obtain damped equations (Ott and Sudan, 1970; Kakutani and Matsuuchi, 1975). In fact, the damping in KdV equations can be either a diffusion term that leads to study

$$u_t + +u_{xxx} + uu_x = \nu u_{xx}, \qquad (5.1)$$

where ν is a positive number analogous to the viscosity, or a zero order term $-\nu u$ in the right-hand side of (5.1). In the first case, we obtain some KdV-Burgers equation that has some smoothing effect in time. In the second case, we have a zero order dissipation term. A non local term would be $\nu \mathcal{F}^{-1}(|\xi|^{2\beta}\hat{u}(\xi))$ for $\beta \in]0,1[$, where $\mathcal{F}(u) = \hat{u}$ denotes the Fourier transform of u.

A first issue concerning damped water waves equations is to estimate the decay rate of the solutions towards the equilibrium 0 when times goes to $+\infty$. For (5.1) the ultimate result is that for initial data $u_0 \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$ the L^2 norm of the solution decays like $t^{-1/4}$ (Amick, Bona, Shonbeck 1989). Energy methods have been developed to handle these problems, as the Shonbeck's splitting method.

Another approach takes part in dynamical systems: the center manifold theory. The aim is to prove the existence of a finite dimensional manifold that is invariant (in a neighborhood of the origin) by the flow of the solutions and that attracts the others trajectories with high speed. Therefore this manifold, and the trajectories therein, monitor the decay rate of the solutions towards the origin. The construction of such a manifold relies on splitting properties of the spectrum of the associated linearized operator. Let us mention the article of Gallay and Wayne for 2D Navier Stokes equations, 2002. Using a suitable change of variables (that moves the continuous spectrum away from the origin), they were able to construct such a manifold in an infinite-dimensional phase space.

Another issue is the understanding of the dynamics for damped-forced water waves equations as

$$u_t + uu_x + u_{xxx} + \nu u = f(x). \qquad (5.2)$$

The dynamical system approach is the *attractor* theory (see Temam, 1997). The equations like (5.2) provide dissipative semi-groups $S(t)$ in some energy spaces. The theory has developed for years and we know that these dynamical systems feature global attractors. A global attractor is a compact subset in the energy space under consideration which is invariant by the flow of the solutions and that attracts all the trajectories when times goes to $+\infty$. Moreover, if we deal with periodic boundary conditions, this global attractor has finite fractal (or Hausdorff) dimension. This dimension depends on the data ν, f.

Actually, the equation (5.2) provides semi-groups either in $L^2(\mathbb{R}), H^1(\mathbb{R})$ or in $H^2(\mathbb{R})$. These three dynamical systems feature global attractors $\mathcal{A}_0, \mathcal{A}_1, \mathcal{A}_2$. For the physics, the attractors describe the permanent regime of the flow. One may wonder if this permanent regime depends on the space we have chosen for the mathematical study. Eventually the last result for this issue establishes that these three attractors are in fact the same set. This property is equivalent to prove the *asymptotical smoothing effect* for the associated semigroup: even if $S(t)$ is not a smoothing operator for finite t, then all solutions converge to a smooth set when t goes to the infinity.

All these results are for subcritical nonlinearities. As we have seen, dissipation provides smoothing at the infinity. Nevertheless damping does not prevent blow up. Let us illuminate this by the following result due to M. Tsutsumi, 1984. The damped Schrödinger equation

$$iu_t + i\nu u + u_{xx} + |u|^{2p}u = 0, \qquad (5.3)$$

features blow up solutions in $H^1(\mathbb{R})$ for $p > 2$, even if all solutions are damped in $L^2(\mathbb{R})$ with exponential speed.

This completes the section of damped-forced water waves equations. We now consider equations that are forced with a random forcing term.

6. Stochastic water waves models

During the modelisation process that led to KdV or Schrödinger equations from Euler equation, we have neglected some low order terms. We now model these terms by a noise and we are led with a new randomly forced dynamical system that reads

$$u_t + u_x + u_{xxx} + uu_x = \gamma\dot{\xi}. \qquad (6.1)$$

Here one may pretend that $\xi(x,t)$ is a Gaussian process with correlations

$$I\!E(\dot{\xi}(x,t)\dot{\xi}(y,s)) = \delta_{x-y}\delta_{t-s}, \qquad (6.2)$$

that is a space-time white noise. The parameter γ is the amplitude of the process. Unfortunately, due to the lack of smoothing effect of KdVor Schrödinger equations, we rather work with a noise that is correlated in space, satisfying

$$I\!E(\dot{\xi}(x,t)\dot{\xi}(y,s)) = c(x-y)\delta_{t-s}; \qquad (6.3)$$

here $c(x-y)$ is some smooth anszatz for δ_{x-y}, defined from some Hilbert-Schmidt kernel K as

$$c(x-y) = \int_{\mathbb{R}} K(x,z)K(y,z)dz.$$

We also consider random perturbation of focusing Schrödinger equation that reads either

$$u_t + iu_{xx} + i|u|^{2p}u = u\dot{\xi}, \qquad (6.4)$$

which represents a multiplicative noise or

$$u_t + iu_{xx} + i|u|^{2p}u = i\gamma\dot{\xi}, \qquad (6.5)$$

which is an additive noise. In the former case, the noise acts as a potential, while in the latter case it represents forcing term. These equations also model the propagation of waves in an inhomogeneous media.

Current researches are in progress to study these stochastic dynamical systems. To begin with, the theory of the initial value problem has to be established in this new context; see for instance de Bouard an Debussche, 2003.

One challenge is to understand the effect of the noise on dynamical properties of the particular solutions described above, for instance the solitary waves for Schrödinger , either in the subcritical case $p < 2$ or in the critical case $p = 2$ and beyond.

Let us describe the results obtained both theoretically and numerically on the influence of the noise on blow up phenomena for generalized Schrödinger equations. Since we are dealing with random process, we are talking about almost sure results.

On the one hand, if the noise is additive and the power supercritical $p > 1$, there are some numerical evidences that a space-time white noise can delay or even prevent the blow up. However, if the noise is not so irregular (as for the correlated in space noise described above) it seems that *any* solution blows up in finite time.

Eventually de Bouard and Debussche proved that for either an additive or a multiplicative noise, any smooth and localized in space initial data gives rise to a trajectory that collapses in arbitrarily small time with a positive probability. This constrats with the deterministic case, where only *particular* initial data could led to blow up trajectories. Actually, the noise enforces that any trajectory has to visit this blow up region, with a positive probability.

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See Also

Central manifolds, stability theory, dissipative dynamical systems of infinite dimensions, KAM theory, waves in fluids, nonlinear Schrödinger equations, scattering theory: asymptotic completness and bound states.

Further reading

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